Sensitivity Analysis for an Optimal Routing Policy in an Ad Hoc Wireless Network

Tara Javidi and Demosthenis Teneketzis
Department of Electrical Engineering and Computer Science
University of Michigan
Ann Arbor, MI 48109
taraj@eecs.umich.edu teneket@eecs.umich.edu

***Abstract*— We examine the sensitivity of optimal routing policies in ad hoc wireless networks with respect to estimation errors in channel quality. We consider an ad hoc wireless network where the wireless links from each node to its neighbors are modeled by a probability distribution describing the local broadcast nature of wireless transmissions. These probability distributions are estimated in real-time. We investigate the impact of estimation errors on the performance of a set of proposed routing policies.**

I. Introduction

Due to the variability and uncertainty in the behavior of the wireless channel, wireless networks should be modeled as stochastic systems. Hence, service provisioning and resource allocation issues (such as admission control, routing, etc.) in wireless networks are best modeled as stochastic scheduling and stochastic control problems, where the wireless links are described by stochastic processes. The statistics of any wireless link depends on the physical channel (additive noise, path loss, shadowing, fading, etc.), the number of users that use the link simultaneously, and the users' transmission strategies. Generally the overall structure and statistical behavior of the system, e.g. the marginal and joint distributions of the processes involved is studied and modeled off-line, while the particular parameters of such models, e.g. mean and covariance, are left to be estimated in a real-time measurement-based fashion. For instance, a single-hop wireless link might be modeled as an independent identically distributed binary symmetric channel, whose transmission error probability p_e is estimated on-line. In such systems the control strategy regulates all communications and can potentially provide information on the statistics of the wireless channels. Hence the estimation problem combined with the control issues should be ideally studied as a stochastic control problem with imperfect information. Stochastic control problems with imperfect information are dual control problems that address joint estimation and control problems [3]. The information state [3] for these problems lies in an infinite dimensional space even when the state space and action space are finite. This feature makes such dual control problems analytically and computationally difficult. An alternative approach is to decouple the control problem from the estimation issues. Such an approach provides a parameter estimation algorithm operating independent of the control decisions and then it feeds the estimated parameters into a controller designed under the perfect information assumption. Following such an approach, service provisioning in wireless networks can be addressed by the following three step procedure: (*i*) Off-line modeling of the overall statistical behavior of the wireless links; (*ii*) specification of the parameters associated with the wireless links, based on real-time measurements and through parameter estimation algorithms; and (*iii*) determination of optimal service provisioning strategies assuming that the results of steps (*i*) and (*ii*) describe the system's true stochastic behavior. As expected, there are errors associated with the estimation techniques used in (*ii*) and the accuracy of the estimated parameters are limited to the employed estimation algorithm's error margin. On the other hand, the optimal control strategy resulting from (*iii*) is guaranteed to be optimal only for the particular values of parameters given by (*ii*). In other words, the performance of such policy deteriorates as the distance between the estimated broadcast model and the true model (estimation error) increases. Hence, it is vital to quantify the loss in performance of the proposed routing strategy with respect to the aforementioned estimation errors.

In this paper we present a sensitivity analysis of a known optimal (with respect to an energy consumption criterion) routing policy in a stochastic ad hoc network.

Our analysis is based on the model and results of [5] and [4]. In [5], the authors investigate a time-invariant network routing problem where a probabilistic model for wireless local broadcasts is used (see Fig. 1). Under the assumption that the transmission probabilities of the local broadcast model for each node are precisely known, the existence of an optimal priority policy with time-invariant indices is shown. As expected, these indices depend on the parameters of the local broadcast model. Furthermore, the authors provide a distributed implementation of the optimal policy. We investigate the sensitivity of this priority policy with respect to errors in the knowledge of the aforementioned transmission probabilities, and analytically determine the impact of errors in the broadcast model on the performance of the optimal policy. We quantify this impact as follows: (*i*) we construct policies $\tilde{\pi}$ and π^* as the optimal solutions to the true model P and the estimated model Q, respectively; and (*ii*) we bound the distance between the performance of the two policies $\tilde{\pi}$ and π^* by a term proportional to the distance between broadcast models P and Q (estimation error). We extend this result to the distributed construction of the decentralized optimal routing strategy.

The remainder of this paper is organized as follows. In Part II, we formulate the problem we analyze. In Part.II.A we first present the formulation of the routing problem in an ad hoc networks with perfect knowledge of broadcast models, provide some useful notation and definitions, and state the result given by [5] on the structure of the optimal routing strategy. In Part II.B we formulate the sensitivity problem, construct examples, and provide a relationship between error and loss of performance. In part III, we briefly discuss the extension of our result to the distributed computation and construction of optimal routing policies in a network with distributed information and decentralized control.

II. Problem Formulation

We first revisit the following problem (Problem ($\mathbf{P_1}$)) formulated in [5], and state the results proved in [5] that are necessary for our analysis. Based on Problem ($\mathbf{P_1}$), we formulate Problem ($\mathbf{P_2}$) which is an investigation of the sensitivity of the optimal routing policy with respect to errors in estimation of the probabilities of successful transmission.

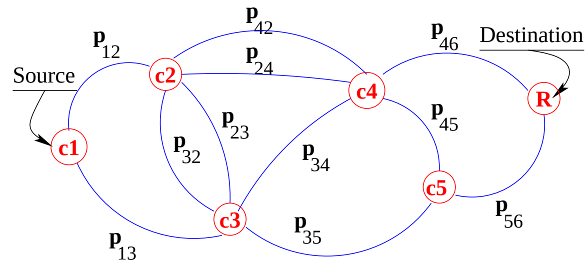

Fig. 1. Ad hoc wireless network with probabilistic local broadcast model

A. *Problem ($\mathbf{P_1}$): Statement and Results*

A.1 Model (**M**): Notations and Preliminaries

We begin by briefly defining notations and stating definitions for the system model, which we refer to as Model (**M**), under consideration. As discussed in [5], Model (**M**) is probabilistic and control is centralized, meaning that the controller has access to all the information available at the network. A description of the elements of this network model is given below.

N is the number of nodes in the network.

$\omega = \{1,\dots,N\}$, the set of all nodes. So $|\omega| = N$.

$S \subseteq \omega$ refers to a state of the system, defined as the set of nodes which have received the message. S_t refers to the state at time t.

We define $\mathcal{S} := \{S : S \subset \omega\}$.

We write $P^i(S'|S)$ to indicate the probability of reaching state S' from state S when choosing i for transmission, $i \in S$. We write $P^i(S|i)$ as shorthand for $P^i(S|\{i\})$.

We refer to $P = \{P^i(S'|S)\}_{i,S',S}$ as the broadcast model.

We define $P_{ij} := \sum_{S:i,j\in S} P^i(S|i)$.

We assume that transmission events at a given node are i.i.d., and transmission events are independent among all nodes.

j is called a *neighbor* of i if $P_{ij} > 0$.

Given the local broadcast model P, $\mathcal{N}_P(i)$ is the set of all neighbors of i, together with i itself. Note that $P_{ij} \neq P_{ji}$ is permitted.

Definition 1 (Increasing Property) Model (**M**) is said to be *increasing* if for any system realization under any policy we have $S_{t_2} \supseteq S_{t_1}, \ \forall t_1, \forall t_2 > t_1$.

Definition 2 (Decoupling Property) Model (**M**) is said to be *decoupled* if transmission success to a set of neighbors from a node at a given time is unaffected by which other nodes already have the message.

We assume that Model (**M**) is both increasing and decoupled.

$R : 2^{\omega} \to \mathbb{R}^{+}$ is the reward function, and $R_i := R(\{i\})$. Also $R_{\max} := \max_{i \in \omega} R_i$.

π is a Markov policy.

We write $\pi(S) = i$ to indicate policy π selects transmission at node i when in state S.

We write $\pi(S) = r$ to indicate policy π retires and receives reward $R(S)$ when in state S. For convenience we write $\pi(S) = r_i$ as shorthand that policy π retires and receives $R_i, i \in S$. In this case, we say that policy π retires and *receives the reward* of node i.

By $\pi(S) \neq i, r_i$, we mean both $\pi(S) \neq i$ and $\pi(S) \neq r_i$.

By $\pi(S) = \tilde{\pi}(S)$, we mean either $\pi(S) = \tilde{\pi}(S) = i$, or $\pi(S) = \tilde{\pi}(S) = r_i$, for some i.

Each transmission from node i incurs a cost of c_i.

We next formulate the centralized version of the stochastic routing problem with time-invariant parameters.

A.2 Statement of Problem ($\mathbf{P_1}$)

<u>Problem ($\mathbf{P_1}$)</u>

We consider the transmission of a single message, from a given initial state S_0 (i.e. a given set of nodes) to a set of destination states, in a wireless ad hoc network of N nodes described by Model (**M**) in which the transition probabilities are given by the broadcast model P. Transmission instances occur at discrete time points. Each transmission from a given node i incurs a fixed cost $c_i > 0$. According to Model (**M**): (*i*) at each transmission instance the transmitting node is chosen by a central controller that always knows the current state of the system (i.e. the set of nodes that have the message); (*ii*) node transmissions are local broadcasts, that is, multiple neighbor nodes may all simultaneously receive the message; (*iii*) given the node chosen to transmit, the probability that a given set of nodes receives the message is known and fixed; (*iv*) The central controller is informed, without any cost, as to which nodes receive the message. Control information flow between the nodes and the controller is considered free of energy and instantaneous in time; (*v*) each transmission event is assumed independent of those before and after; (*vi*) a reward function R is specified. At any instance, the central controller can terminate the transmission process or choose to continue transmitting. The objective is to choose: (*i*) the node to transmit at each transmission instance, and (*ii*) the instance to terminate the transmission process, to maximize over all Markov policies,

$$J_P^{\pi}(S_0) = E\left\{ R(S_f) - \sum_{t=1}^{\tau-1} c_{i(t)} \right\},$$

where π is the transmission/termination policy the controller follows, τ is the time when the transmission process is terminated under policy π, S_f is the state at τ, and $i(t)$ is the node chosen by the transmission policy at time t, $J_P^{\pi}(S)$ is the expected reward when starting in state S under policy π under local broadcast model P. Restriction to Markov policies does not entail any loss of optimality because Problem ($\mathbf{P_1}$) is a stochastic control problem with perfect observations [3]. Notice that Model (**M**)and Problem ($\mathbf{P_1}$) can also address the optimal routing and hop selection in networks where nodes have a set of transmission powers to select from (see [5]).

Mathematically, Problem ($\mathbf{P_1}$) is parameterized by a tuple $(N, P, \underline{c}, R)$.

A.3 Results: Optimal Routing in Problem ($\mathbf{P_1}$)

In this section, we summarize the results in [5] that are relevant to our work. To present these results we need the following definitions:

Definition 3: A Markov policy π is a priority policy if there is a strict priority ordering of the nodes s.t. $\forall i \in \omega$ we have $\pi(S \cup \{i\}) = \pi(\{i\}) = i$ or $r_i, \forall S \subset \Omega_i^{\pi}$, where Ω_i^{π} is the set of nodes of priority lower than i.

Definition 4: For priority policy π, we write $i >_{\pi} j$ when i has higher priority than j under π.

Now we state the following facts from [5]

Fact. 1: For priority policy π we have $J_P^{\pi}(S) = J_P^{\pi}(\{\pi(S)\}) = J_P^{\pi}(\{i\})$, when $i >_{\pi} j$ for $\forall j \in S - \{i\}$.

This fact holds by the decoupling property and the definition of a priority policy.

Fact. 2: There is an optimal Markov policy π^* for Problem ($\mathbf{P_1}$) which is a priority policy whose expected reward has the following property:

$$J_P^{\pi^*}(S) = \max_{i \in S} J_P^{\pi^*}(\{i\})$$

Fact. 3: Under the optimal Markov policy π^* the expected reward for each node i defines an index $J_P^{\pi^*}(\{i\})$. This index, in turn, defines the optimal ordering of the nodes and the actions taken at these nodes; i.e.

$$J_P^{\pi^*}(\{i\}) > J_P^{\pi^*}(\{j\}) \implies i >_{\pi^*} j$$
$$i >_{\pi^*} j \implies J_P^{\pi^*}(\{i\}) \geq J_P^{\pi^*}(\{j\})$$

All these facts which are proved in [5] establish an optimal routing (priority ordering of nodes) under the assumption that P is fixed and known. In reality P is not known but has to be estimated (see discussion in [5]). Hence, the presence of estimation errors raises the important issue of the sensitivity of the results in [5] with respect to (small) variations in P. This motivates the sensitivity analysis presented in this paper.

B. Sensitivity Analysis: Problem Formulation

Consider Problem $(\mathbf{P_1})$ associated with two sets of parameters, $(N,P,\underline{\mathbf{c}},R)$ and $(N,Q,\underline{\mathbf{c}},R)$, describing the true and estimated models of the system, respectively. According to the results given in Section II-A.3 there exists an index policy π^* which is an optimal routing policy for Problem $(\mathbf{P_1})$ with parameters $(N,P,\underline{\mathbf{c}},R)$. At the same time the optimal solution to the estimated model, $(N,Q,\underline{\mathbf{c}},R)$, is an index policy $\tilde{\pi}$ that is not optimal for the true model $(N,P,\underline{\mathbf{c}},R)$, in general. Policy $\tilde{\pi}$ is applied to the system with the true broadcast model (distribution) P. We are interested in: (i) Determination/quantifying the difference between the performance of policy $\tilde{\pi}$ in such a system and the best possible performance, achieved by π^*. (ii) Relating the aforementioned difference to a quantity describing the error in estimation of the (true) broadcast model.

To quantify the difference specified in (i) we define an appropriate metric on the space of all routing policies which captures the maximum loss of performance when a policy is applied to the system with model P. We define *the distance between* π_1 *and* π_2 *at state* S *in the context of the distribution* P as

$$d_P(\pi_1,\pi_2,S) := |J_P^{\pi_1}(S) - J_P^{\pi_2}(S)|.$$

We define *the distance between policies* π_1 *and* π_2 *in the context of distribution* P as

$$d_P(\pi_1,\pi_2) := \max_S |J_P^{\pi_1}(S) - J_P^{\pi_2}(S)|$$

To relate the difference specified in (i) to the estimation error in the (true) broadcast model we first quantify this error by defining a distance measure between the true broadcast model P and the estimated model Q. We use the total variation metric for this purpose (see [6], and [1]). The total variation distance between two local broadcast models, P and Q, describing the probabilities of transmission success for node i, is defined as

$$\sigma(P_i,Q_i) = \sup_{A\subset S}\left|\sum_{S'\in A}\left(P^i(S'|i) - Q^i(S'|i)\right)\right|.$$

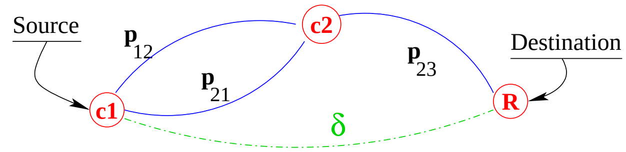

Fig. 2. Example 1.

Based on the above we reformulate the following sensitivity analysis problem.

Problem $(\mathbf{P_2})$

Consider Problem $(\mathbf{P_1})$ for two sets of parameters, $(N,P,\underline{\mathbf{c}},R)$ and $(N,Q,\underline{\mathbf{c}},R)$, describing the true and estimated models of the system, respectively. Let π^* be an optimal routing policy for Problem $(\mathbf{P_1})$ with parameters $(N,P,\underline{\mathbf{c}},R)$, and $\tilde{\pi}$ be an optimal routing policy for Problem $(\mathbf{P_1})$ with parameters $(N,Q,\underline{\mathbf{c}},R)$. The objective is to determine the distance between policies $\tilde{\pi}$ and π^* in the context of P, and relate this distance, if possible, to the total variation distance between the estimated model Q and the true system model P, i.e. $\max_i \sigma(P_i,Q_i)$.

C. Sensitivity Analysis: Results

In Section II-C.1 we show that, in general, the optimal routing can be extremely sensitive to estimation error, i.e. there exist scenarios where a small error in estimation can deteriorate the performance of the constructed priority policy unboundedly. In Section II-C.2 we assume that the percentile of error is bounded. We show that under such assumption the loss of performance is bounded by a term proportional to the estimation error.

C.1 Example

Consider the simple network given by Fig. 2. We assume that the probabilities of successful transmission are given by the true model P and the estimated values of these probabilities are given by model Q. The value of these transmission probabilities are $P^1(\{2,3\}|1) = Q^1(\{2,3\}|1) = 0$, $P^1(\{2\}|1) = Q^1(\{2\}|1) = p$, $P^2(\{1\}|2) = Q^2(\{1\}|2) = p$, $P^2(\{1,3\}|2) = Q^2(\{1,3\}|2) = 0$, and finally $P^2(\{3\}|2) = Q^2(\{3\}|2) = p$. Furthermore, we assume $P^1(\{3\}|1) = 0$ while $Q^1(\{3\}|1) = \delta$. Assume that node i has transmission costs c_i, $i = 1,2$; Reward is zero for the first two nodes, and is equal to R at the destination; And the cost of transmission at node 2 is much larger than the cost of transmission at node 1, i.e. $c_2 \gg c_1$.

In this example the total variation distance between the two broadcast models P and Q, i.e. $\max_i \sigma(P_i, Q_i)$, is δ. Policies π^* and $\tilde{\pi}$ are the only candidates for the optimal policy in this example and under these policies we have $1 <_{\pi^*} 2 <_{\pi^*} 3$ and $2 <_{\tilde{\pi}} 1 <_{\tilde{\pi}} 3$. The distance between these two policies in the context of P is infinite, since $J_P^{\tilde{\pi}}(\{1\}) = \infty$. Here we show that even for a small distance between models P and Q, i.e. small δ, policy $\tilde{\pi}$ can be selected as the optimal policy (due to its optimality in the context of Q), hence causing an unbounded loss of performance. To prove this, we write the expected rewards:

$$J_Q^{\pi^*}(\{2\}) = R - \frac{c_2}{p} \qquad J_Q^{\pi^*}(\{1\}) = R - \frac{c_1 + c_2}{\delta + p}$$
$$J_Q^{\tilde{\pi}}(\{1\}) = R - \frac{c_1}{\delta} \qquad J_Q^{\tilde{\pi}}(\{2\}) = R - \frac{c_2}{2p} - \frac{c_1}{2\delta}$$

Since $c_2 \gg c_1$, there exists a (small) δ for which $J_Q^{\pi^*}(\{i\}) \leq J_Q^{\tilde{\pi}}(\{i\})$, $i = 1, 2$, i.e. $\tilde{\pi}$ is selected as the optimal policy. This example illustrates that, in general, a small error in channel estimation can cause an unbounded loss of performance.

C.2 Analysis of Problem ($\mathbf{P_2}$)

In this section our goal is to bound the distance between two policies $\tilde{\pi}$ and π^* by a term proportional to the distance between broadcast models P and Q. As illustrated in Example II-C.1, this is not possible in general. That is why we make the following assumption on the nature of the estimation error.

Assumption 1: For any node i, there exists $M_i < \infty$ such that $Q^i(S|i) - P^i(S|i) \leq M_i P^i(S|i)$.

Intuitively, Assumption 1 has two significant implications. First, it implies that the network topology under the estimated broadcast model Q does not contain links which do not really exist, i.e. $\forall i \in \{1, 2, \cdots, N\}$, $\mathcal{N}_Q(i) \subset \mathcal{N}_P(i)$. Second, the assumption implies that, whenever there is a link between nodes i and j, i.e. $P_{ij} > 0$, there is a finite bound M_i which specifies the maximum percentile of error in the estimation of the quality of links connected to node i.

Under the above assumption we prove the following theorem which summarizes the main result of this paper. For the proof of Theorem 1, see [2]. Our approach is inspired by the general framework proposed in [6].

Theorem 1: Under Assumption 1, we have

$$d_P(\pi^*, \tilde{\pi}) \leq K \max_j \sigma(P_j, Q_j),$$

where $K = \sum_{j=1}^{N} \frac{R_{\max}(R_{\max} - R_j)(M_j + 2)}{c_j}$.

III. Distributed Computation of the Optimal Policy

As mentioned in the introduction, the key feature of an ad hoc network is the absence of central control or computation unit. This feature underlines the importance of providing a distributed algorithm for the computation and implementation of an optimal policy. The authors in [5] provide three algorithms according to which each node can compute its optimal local routing decisions based only on the local information and local communication with its neighbors. Notice that in these distributed algorithms, each node requires information about its own local broadcast model. This implies that, in practice, it is sufficient for each node to estimate the broadcast model locally. Consequently, it is vital to study the sensitivity of the optimal local decisions in the presence of error. In [5], it is shown that almost in all practical scenarios under the proposed algorithms, the local routing decisions converges to a stationary index policy equivalent to the index policy described in Section II-A.3. This implies that, given a sufficiently long time horizon, all of these algorithms demonstrate identical performance loss in the presence of estimation error in broadcast model. And this loss is related to the estimation error in the same fashion as that of the centralized index policy discussed in Section II-C.2, Theorem 1.

Acknowledgment

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