

ECE257A Fall 2007

Solution for Problem Set 2

November 23, 2007

Problem 1

$$\|\alpha x_1 + (1 - \alpha)x_2\| \leq \|\alpha x_1\| + \|(1 - \alpha)x_2\| = \alpha\|x_1\| + (1 - \alpha)\|x_2\|$$

$$d(x_1, x_2) = \|x_1 - x_2\| = \|A(x_1, x_2)\|$$

where A is the linear function representing the subtraction. Now, we show that given a linear map A , and a convex function g , function $g_A(x) := g(Ax)$ is convex.

$$\begin{aligned} g_A(\alpha x_1 + (1 - \alpha)x_2) &= g(A(\alpha x_1 + (1 - \alpha)x_2)) = g(\alpha(A(x_1)) + (1 - \alpha)(A(x_2))) \\ &\leq \alpha g(A(x_1)) + (1 - \alpha)g(A(x_2)) = \alpha g_A(x_1) + (1 - \alpha)g_A(x_2) \end{aligned} \quad (1)$$

Now using (1), convexity of Euclidean distance is a direct consequence of convexity of Euclidean norm.

Problem 3

a) The Chain is irreducible, since for $\forall j$

$$0 < \max \{P^{|j-k|}(k, j), P^{|j-k|+1}(k, j)\} = \begin{cases} \prod_{i=k}^{j-1} a_i & \text{if } k < j \\ (1 - a_k)\prod_{i=0}^{k-1} a_i & \text{if } k > j \end{cases}$$

b) Since the chain is irreducible, if any state is identified as recurrent or positive recurrent, all states are. In other words, it is sufficient to determine the conditions for recurrence and positive recurrence for state "0."

Necessary and sufficient condition for recurrence:

$$\prod_{i=0}^{\infty} a_i = 0.$$

Necessary and sufficient condition for positive recurrence:

$$\prod_{i=0}^{\infty} a_i = 0 \text{ and } \sum_{m=1}^{\infty} \prod_{i=0}^{m-1} a_i < \infty$$

Problem 4

a) We have the following recursion:

$$P(\text{win}|X_0 = i) = pP(\text{win}|X_0 = i + 1) + (1 - p)P(\text{win}|X_0 = i - 1)$$

This simplifies to

$$P(\text{win}|X_0 = i) - P(\text{win}|X_0 = i - 1) = \frac{p}{1 - p} (P(\text{win}|X_0 = i + 1) - P(\text{win}|X_0 = i))$$

On the other hand, we have the boundary conditions:

$$P(\text{win}|X_0 = 0) = 0 \text{ and } P(\text{win}|X_0 = N)$$

From this and $p = 1/2$, we have that

$$P(\text{win}|X_0 = i) = i/N.$$

b) Similarly, one step recursion gives the expected number of plays before the end of the game. Let T be the stopping time associated with the end of the game.

$$E(T|X_0 = i) = pE(T|X_0 = i + 1) + (1 - p)E(T|X_0 = i - 1) + 1$$

with boundary conditions:

$$E(T|X_0 = 0) = E(T|X_0 = N) = 0$$

Let $p = 1/2$. We get

$$E(T|X_0 = 1) = (N - 1).$$

Problem 5

Postponed to the next Problem set.