ECE257A Fall 2007
Solution for Problem Set 3

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Problem 1
Let A be m x n matrix.
Writing the problem in the form we saw in class will be:

$$\min b^T x$$

subject to

$$-Ax + c \leq 0$$

The Dual problem is $\max_{\lambda \geq 0} q(\lambda)$ where

$$\begin{aligned} q(\lambda) &= \inf_{x \geq 0} \left\{ b^T x + < -Ax + c, \lambda > \right\} \\ &= \inf_{x \geq 0} \left\{ \sum_{j=1}^{n} b_j x_j - \sum_{i=1}^{m} \lambda_i [(\sum_{j=1}^{n} a_{ij} x_j) - c_i] \right\} \\ &= \inf_{x \geq 0} \left\{ \sum_{j=1}^{n} (b_j - \sum_{i=1}^{m} \lambda_i a_{ij}) x_j + \sum_{i=1}^{m} \lambda_i c_i \right\} \end{aligned}$$

If $b_j - \sum_{i=1}^{m} \lambda_i a_{ij} \geq 0$, $\forall j = 1, 2, \ldots, n$, then the inf is attained for $x = 0$ and $q(\lambda) = \sum_{i=1}^{m} \lambda_i c_i$.

On the other hand, if $b_j - \sum_{i=1}^{m} \lambda_i a_{ij} < 0$ for some j, then $q(\lambda) = -\infty$ (the larger the x_j the more negative $q(\lambda)$). Since we are interested in choosing λ to maximize $q(\lambda)$, we need to avoid such a case. In other words, we need to constraint the choice of λ such that $sum_{i=1}^{m} \lambda_i a_{ij} \leq b_j$ for all j.

Thus the dual problem is

$$\max_{\lambda \geq 0} q(\lambda) = \max_{\lambda \geq 0} \sum_{i=1}^{m} \lambda_i c_i$$

subject to

$$\sum_{i=1}^{m} \lambda_i a_{ij} \leq b_j, \forall j = 1, 2, \ldots, n$$

Problem 2

Let f_1 and f_2 be convex functions i.e. for any a_1, a_2, $f(a_1x_1 + a_2x_2) \leq a_1 f(x_1) + a_2 f(x_2)$.

Let $g(x) = \max(f_1(x), f_2(x))$. g is convex because

$$\begin{aligned} g(a_1x_1 + a_2x_2) &= \max(f_1(a_1x_1 + a_2x_2), f_2(a_1x_1 + a_2x_2)) \\ &= f_i(a_1x_1 + a_2x_2), \end{aligned}$$

where $i = 1$ if $f_1(a_1x_1 + a_2x_2) \geq f_2(a_1x_1 + a_2x_2)$ and $i = 2$ otherwise. Hence,

$$\begin{aligned} g(a_1x_1 + a_2x_2) &= f_i(a_1x_1 + a_2x_2) \\ &\leq a_1 f_i(x_1) + a_2 f_i(x_2) \\ &\leq a_1 g(x_1) + a_2 g(x_2), \text{ by definition of } g. \end{aligned}$$

However, $\min(f_1, f_2)$ is not convex. Counter example: $f_1(x) = x$ and $f_2(x) = -x$. $h(x) = \min(f_1(x), f_2(x)) = -|x|$ and h is not convex.

Problem 3

The dual problem is $\max_{\mu \geq 0} q(\mu)$ where

$$q(\mu) = \inf_{x \geq 0} \left\{ \frac{1}{2} x^T Q x - b^T x + \mu^T (Ax - c) \right\}.$$

Setting $\frac{\partial}{\partial x} = 0$ and assume that Q is symmetric, the *inf* is attained for $x = -Q^{-1}(A^T\mu - b)$. We can go on and provide the general solution for $\max_{\mu \geq 0} q(\mu)$. However, since the question assumes simple Q, $b = 0$, and $A = [1, 1]$, the problem becomes

$$\min_x x_1^2 + x_2^2/2$$

subject to

$$x_1 + x_2 \leq c$$

The dual is $\max_{\mu \geq 0} q(\mu)$ where

$$\begin{aligned} q(\mu) &= \inf_{x \geq 0} \left\{ x_1^2 + x_2^2/2 + \mu(x_1 + x_2 - c) \right\} \\ &= -3\mu^2/4 - \mu c. \end{aligned}$$

$q(\mu)$ is maximized when $\mu = \begin{cases} -2c/3, & \text{if } c < 0, \\ 0, & \text{otherwise} \end{cases}$ and its maximum value is $\begin{cases} c^2/3, & \text{if } c < 0, \\ 0, & \text{otherwise} \end{cases}$.

Problem 4

Follow the similar proof as in the class.

Proof: ($\Rightarrow$) Assume x^* solves the network utility problem for $g(x) = \sum_i p_i f_\alpha(x_i)$ where $f_\alpha(x_i) := x_i^{1-\alpha}/(1-\alpha)$.

$$\nabla g^T|_{x^*} = (p_1 x_1^{-\alpha}, \ldots, p_m x_m^{-\alpha}) = \mu^T A^T$$

$$(p_1 x_1^{-\alpha}, \ldots, p_m x_m^{-\alpha})(x - x^*) = \mu^T A^T (x - x^*) \leq 0.$$

Thus, x^* is (p, α)-proportionally fair.

Proof: $(\Leftarrow)$ Assume x^* is (p, α)-proportionally fair. We have

$$\nabla g^T|_{x^*}(x - x^*) = \mu^T A^T (x - x^*) \leq 0.$$

and since g is concave, $\nabla^2 g$ is negative semi-definite. We have x^* is optimal.

Problem 5

Proof (by contradictiction):

Assume $\underline{r} = (r_1, \ldots, r_n)$ is maxmin fair but $\exists j$ such that $r_j > \min_k r_k = r_i$ (i attains the minimum on the right hand side of this inequality).

From the solidarity of the feasibility region, we have that $\exists \epsilon$ such that for $\forall \alpha \leq \epsilon$ and $\forall j$, $\exists \delta > 0, (\delta \leq \epsilon)$ such that $\forall k, \underline{r} - \alpha e_j + \delta e_k$ is also feasible.

Let $\alpha = \min\{\epsilon, r_j - r_i\}$ and $k = i$, to get that $\exists \delta < \epsilon$ such that $\underline{r'} = \underline{r} - \alpha e_j + \delta e_i$ is feasible. This means that rate i is increased with only decrease in rate j which was initially assigned greater rate. This is a contradiction with $\underline{r}$ being max-min fair solution.

To arrive at uniqueness, one needs to notice that no max-min fair solution can be in the interior of feasibility region. In other words, the max-min fair solution in a solidarity feasible region is the unique intersection of line $x_i = x_k$ and the boundary of the feasibility region.