The Theory and Design of Arbitrary-Length Cosine-Modulated Filter Banks and Wavelets, Satisfying Perfect Reconstruction

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***Abstract*—It is well known that FIR filter banks that satisfy the perfect-reconstruction (PR) property can be obtained by cosine modulation of a linear-phase prototype filter of length $N = 2mM$, where M is the number of channels. In this paper, we present a PR cosine-modulated filter bank where the length of the prototype filter is arbitrary. The design is formulated as a quadratic-constrained least-squares optimization problem, where the optimized parameters are the prototype filter coefficients. Additional regularity conditions are imposed on the filter bank to obtain the cosine-modulated orthonormal bases of compactly supported wavelets. Design examples are given.**

I. Introduction

Digital filter banks are used in a number of communication applications such as subband coders for speech signals [1]–[3], frequency domain speech scramblers [4], and image coding [5]–[7]. Fig. 1(a) illustrates a typical M-channel maximally decimated parallel filter bank, where $H_k(z)$ and $F_k(z), 0 \leq k \leq M-1$, are analysis and synthesis filters, respectively (only finite impulse response (FIR) filters are considered in this paper). The analysis filters $H_k(z)$ channelize the input signal $x(n)$ into M subband signals, which are downsampled (decimated) by a factor M. In speech compression and transmission applications [1]–[4], these M subband signals are encoded and transmitted. At the receiving end, the M subband signals are decoded, interpolated, and recombined using a set of synthesis filters $F_k(z)$. The decimator, which decreases the sampling rate of the signal, and the interpolator, which increases the sampling rate of the signals, are denoted by the boxes that are down-arrowed and up-arrowed, respectively, as shown in the figure [2]. The theory for the perfect reconstruction (PR) filter bank has recently been established [8]–[13].

Recently, the PR cosine-modulated filter bank has emerged as an attractive choice of filter bank with respect to implementation cost and design ease [14]–[19]. The impulse responses of the analysis and synthesis filters $h_k(n)$ and $f_k(n)$ are cosine-modulated versions of the prototype filter $h(n)$ [15]. In other words, we have (1), which appears at the bottom of the next page, where N is the length of $h(n)$. It is shown in [15] that the $2M$ polyphase components of the prototype filter $H(z)$ can be grouped into M power-complementary pairs, where each pair is implemented as a two-channel lossless lattice filter bank [13], [15]. The lattice coefficients are optimized to minimize the stopband attenuation of the prototype filter. The design example in [15] for a 17-channel PR cosine-modulated filter bank with stopband attenuation of 40 dB has been presented.

In [15], the length of the prototype filter $H(z)$ is assumed to be an even multiple of M, i.e., $N = 2mM$, where M is the number of channels. For large M, the choice of the length of $H(z)$ becomes quite restrictive. Hence, the following question arises: Do PR cosine-modulated filter banks exist where the filter length is arbitrary, i.e.

$$N = 2mM + m_1, \qquad 1 \leq m_1 \leq 2M - 1? \tag{2}$$

In this paper, we show that the answer is affirmative. Section II discusses the different types of cosine modulation (i.e., Type II and IV) and describes their relationship. Section III derives the PR conditions for the cosine-modulated filter bank with arbitrary length (as in (2)). It turns out that the necessary and sufficient conditions for the arbitrary length case, in terms of the polyphase functions of the prototype filter, are the same as those in [15]. Moreover, these filter banks can be parameterized by pairs of two-channel paraunitary filter banks [13], [15]. Section IV presents the corresponding lattice structure for the paraunitary cosine-modulated filter bank with arbitrary length. The design problem is formulated using the quadratic-constrained least squares method [25], and PR cosine-modulated filter banks with high stopband attenuation can be easily obtained. Several examples (16 channels of length 256, eight channels of lengths 95, and four channels of length 54) are given to demonstrate the generality and robustness of the proposed method.

The theory of wavelets [20]–[24] has been developed independent of the filter bank theory. Wavelets are used as a signal analysis method [20] and as a mathematical tool for constructing original bases for many function spaces [21]–[23]. The work on multiresolution applications [22], [23] has led to an interesting connection between wavelets and filter banks. In particular, it is shown [24] that filter banks, which satisfy regularity conditions, can be used to generate orthornormal bases of compactly supported wavelets. Section VI of the paper imposes additional constraints on the frequency responses

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of the filters of a PR cosine-modulated filter bank in order to obtain compactly supported (cosine-modulated) wavelets. The goal is to obtain a cosine modulated PR filter bank where its lowpass filter $H_0(z)$ is maximally regular. In other words, the filter $H_0(z)$ has a specified number of zeros at $\omega = 2k\pi/M, 1 \le k \le M-1$ [24].

Notations: Bold-faced letters indicate vectors and matrices. Superscript t denotes transposition and the tilde accent on a function $\boldsymbol{F}(z)$ is defined such that $\tilde{\boldsymbol{F}}(z) = \boldsymbol{F}_*^t(z^{-1}), \forall z$, where the asterisk $(*)$ subscript denotes the conjugation of coefficients. Moreover, $\boldsymbol{I}_M$ stands for the $M \times M$ identity matrix and

$$\boldsymbol{J}_M = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{M\times M}.$$

$[\boldsymbol{C}]_{k,\ell}$ denotes the (k,ℓ) element of $\boldsymbol{C}$. $W_M = e^{-j2\pi/M}$.

II. Different Types of Cosine-Modulation

There are four types of discrete-cosine-transform (p. 486 of [26]):

$$\begin{aligned}
&\text{DCT Type I:} && \sqrt{\frac{2}{M}}\left[c_i c_k \cos\left(\frac{ik\pi}{M}\right)\right], \quad i,k=0,\cdots,M\\
&\text{DCT Type II:} && \sqrt{\frac{2}{M}}\left[c_i \cos\left(\frac{i(k+\frac{1}{2})\pi}{M}\right)\right]\\
& && i,k=0,\cdots,M-1\\
&\text{DCT Type III:} && \sqrt{\frac{2}{M}}\left[c_k \cos\left(\frac{k(i+\frac{1}{2})\pi}{M}\right)\right]\\
& && i,k=0,\cdots,M-1\\
&\text{DCT Type IV:} && \sqrt{\frac{2}{M}}\left[\cos\left(\left(i+\frac{1}{2}\right)\frac{(k+\frac{1}{2})\pi}{M}\right)\right]\\
& && i,k=0,\cdots,M-1 \qquad (3)
\end{aligned}$$

where

$$c_i = \begin{cases} 1, & \text{if } i \ne 0 \text{ or } M \\ 1/\sqrt{2}, & \text{if } i = 0 \text{ or } M. \end{cases}$$

In cosine-modulated filter bank, only Type-II and Type-IV modulations are used. Why can't one use Type I and III modulation? To answer this question, consider the following Type-I cosine-modulated filter bank:

$$h_k(n) = 2h(n)c_i c_k \cos\left(\frac{ik\pi}{M}\right)$$

where $h(n)$ is a prototype filter with cut-off frequency at $\pi/2M$. The corresponding z-domain relation is

$$H_k(z) = H(ze^{-jk\pi/M}) + H(ze^{jk\pi/M}).$$

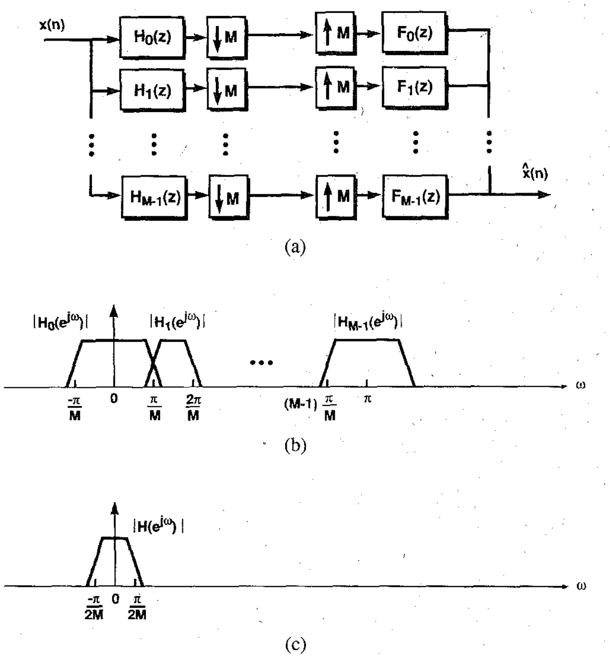


Fig. 1. (a) M-channel maximally decimated parallel filter bank; (b) typical ideal responses of the analysis filters $H_k(z)$; (c) typical ideal response of the prototype filter $H(z)$.

Fig. 2 shows the frequency responses of $H(z)$ and $H_k(z)$ for an eight-channel Type-1 cosine-modulated filter bank. Notice that there are nine filters in the frequency range $0 \le \omega \le \pi$. The bandwidths of $H_0(z)$ and $H_8(z)$ are half of that of the other bandpass filters. Notice that when $k = 0$ in the above equation, then the lowpass filter is $H_0(z) = 2H(z)$. This is not a practical system. In summary, Type-I and Type-III cosine-modulations are not usually used in the construction of cosine-modulated filter bank.

Given that the prototype filter $H(z)$ has the frequency response as in Fig. 1(c), one can verify that only Type II and IV cosine-modulations can produce filter banks that cover all frequency in the range $0 \le \omega \le \pi$, and the filter's bandwidths are the same. These two types are related as follows:

Lemma 1: Let $H(z), H_k(z)$, and $F_k(z)$ be the prototype, analysis, and synthesis filters, respectively, of a Type IV PR cosine-modulated filter bank, i.e.

$$\begin{aligned}
h_k(n) &= 2h(n)\cos\left((2k+1)\frac{\pi}{2M}\left(n-\frac{N-1}{2}\right)+(-1)^k\frac{\pi}{4}\right)\\
f_k(n) &= 2h(n)\cos\left((2k+1)\frac{\pi}{2M}\left(n-\frac{N-1}{2}\right)-(-1)^k\frac{\pi}{4}\right)
\end{aligned} \qquad (4)$$

$$\begin{cases} h_k(n) = 2h(n)\cos\left((2k+1)\dfrac{\pi}{2M}\left(n-\dfrac{N-1}{2}\right)+(-1)^k\dfrac{\pi}{4}\right), \\ f_k(n) = 2h(n)\cos\left((2k+1)\dfrac{\pi}{2M}\left(n-\dfrac{N-1}{2}\right)-(-1)^k\dfrac{\pi}{4}\right), \end{cases}, \quad \begin{cases} 0 \le n \le N-1 \\ 0 \le k \le M-1. \end{cases} \qquad (1)$$

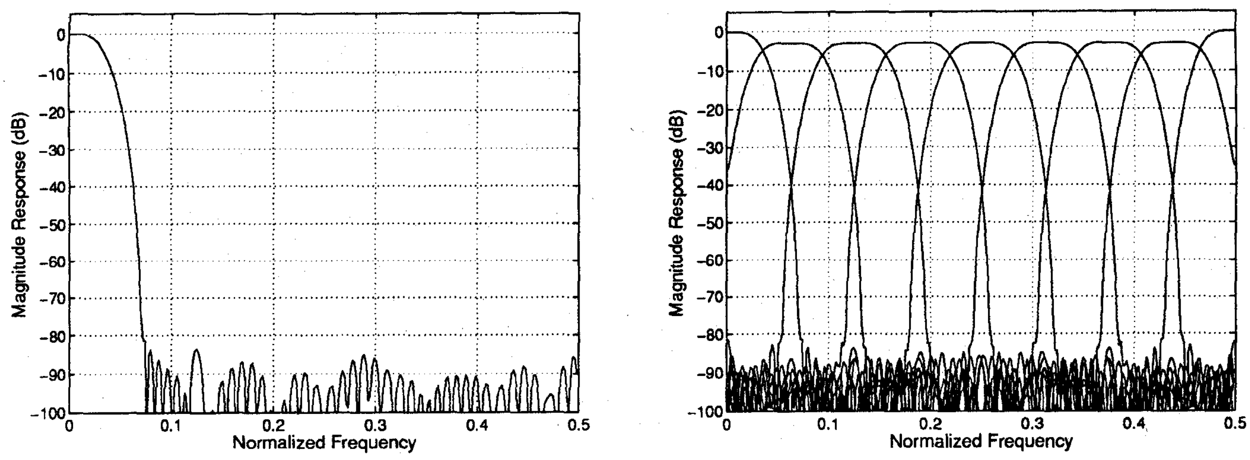


Fig. 2. Type-I cosine-modulated filter bank: (a) $|H(e^{jw})|$ and (b) $|H_k(e^{jw})|$.

then the following filters are the analysis and synthesis filters of Type II PR cosine-modulated filter bank $\hat{N} = N + 1$:

$$\hat{h}_k(n) = 2\hat{h}(n)\cos\left((2k+1)\frac{\pi}{2M}\left(n - \frac{\hat{N}}{2}\right) + (-1)^k\frac{\pi}{4}\right)$$
$$\hat{f}_k(n) = 2\hat{h}(n)\cos\left((2k+1)\frac{\pi}{2M}\left(n - \frac{\hat{N}}{2}\right) - (-1)^k\frac{\pi}{4}\right). \quad (6)$$

Proof: In the frequency domain, $H_k(z)$ and $\hat{H}_k(z)$ are

$$H_k(z) = a_k c_k H(zW^{(k+(1/2))}) + a_k^* c_k^* H(zW^{-(k+(1/2))})$$
$$\hat{H}_k(z) = \hat{a}_k \hat{c}_k \hat{H}(zW^{(k+(1/2))}) + \hat{a}_k^* \hat{c}_k^* \hat{H}(zW^{-(k+(1/2))}) \quad (7)$$

where $W = e^{-j\pi/M}, \theta_k = (-1)^k(\pi/4)$, and

$$\begin{cases} a_k = e^{j\theta_k}, \\ c_k = W^{(k+(1/2))(N-1)/2}, \end{cases}$$
$$\begin{cases} \hat{a}_k = e^{j\theta_k} = a_k, \\ \hat{c}_k = W^{(k+(1/2))(\hat{N}/2))} = c_k W^{(k+(1/2))}. \end{cases} \quad (8)$$

Substituting (5) and (8) into (7), one can verify that

$$\hat{H}_k = z^{-1}H_k(z) \quad (9)$$

(and similarly, $\hat{F}_k = z^{-1}F_k(z)$). Moreover, the overall and aliasing functions $\hat{T}_k(z)$ are

$$\hat{T}_k(z) = \sum_{\ell=0}^{M-1} \hat{F}_\ell(z)\hat{H}_\ell(zW_M^k) = z^{-2}\sum_{\ell=0}^{M-1} F_\ell(z)H_\ell(zW_M^k)$$
$$= z^{-2}T_k(z). \quad (10)$$

In summary, given the prototype filter $H(z)$ of a Type IV PR cosine-modulated filter bank, one can obtain a Type II PR cosine-modulated filter bank by (5) and (6).

III. The Arbitrary-Length PR Cosine-Modulated Filter Bank

Consider the filter bank in Fig. 1(a), where the ideal frequency responses of the filters $H_k(z)$ are shown in Fig. 1(b). Let $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$ be the polyphase component matrices of the analysis and synthesis filter banks, and let $H(z) = \Sigma_{\ell=0}^{2M-1} z^{-\ell}G_\ell(z^{2M})$ be the linear-phase prototype filter of length $N = 2mM + m_1$, where $G_\ell(z)$ are the polyphase filters of $H(z)$. $G_k(z)$ satisfies the following conditions [12]:

$$\tilde{G}_k(z) = z^m \begin{cases} G_{m_1-1-k}(z), & k \le m_1 - 1 \\ z^{-1}G_{2M+m_1-1-k}(z), & k > m_1. \end{cases} \quad (11)$$

It is shown in [15] that $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$ can be expressed in terms of $G_\ell(z)$ and $\hat{\boldsymbol{C}}$ as follows:

$$\boldsymbol{E}(z) = \hat{\boldsymbol{C}}\begin{pmatrix} \boldsymbol{g}_0(-z^2) \\ z^{-1}\boldsymbol{g}_1(-z^2) \end{pmatrix}$$
$$\boldsymbol{R}(z) = (z^{-1}\tilde{\boldsymbol{g}}_0(-z^2) \quad \tilde{\boldsymbol{g}}_1(-z^2))\hat{\boldsymbol{C}}^T \quad (12)$$

where

$$\boldsymbol{g}_0(z) = \mathrm{diag}\,(G_0(z) \quad G_1(z) \quad \cdots \quad G_{M-1}(z))$$
$$\boldsymbol{g}_1(z) = \mathrm{diag}\,(G_M(z) \quad G_{M+1}(z) \quad \cdots \quad G_{2M-1}(z) \quad (13)$$
$$[\hat{\boldsymbol{C}}]_{k,\ell} = 2\cos\left((2k+1)\frac{\pi}{2M}\left(\ell - \frac{N-1}{2}\right) + (-1)^k\frac{\pi}{4}\right). \quad (14)$$

Using the above $\boldsymbol{E}(z)$ and $\boldsymbol{R}(z)$, one obtains the expression for $\boldsymbol{P}(z) \triangleq \boldsymbol{R}(z)\boldsymbol{E}(z)$ as follows:

$$\boldsymbol{P}(z) = (z^{-1}\tilde{\boldsymbol{g}}_0(-z^2) \quad \tilde{\boldsymbol{g}}_1(-z^2))\hat{\boldsymbol{C}}^T\hat{\boldsymbol{C}}\begin{pmatrix} \boldsymbol{g}_0(-z^2) \\ z^{-1}\boldsymbol{g}_1(-z^2) \end{pmatrix}. \quad (15)$$

For a PR filter bank, $\boldsymbol{P}(z)$ must have the form [10]

$$\boldsymbol{P}(z) = \begin{pmatrix} \boldsymbol{0} & z^{-\nu}\boldsymbol{I}_1 \\ z^{-(\nu+1)}\boldsymbol{I}_2 & \boldsymbol{0} \end{pmatrix} \quad (16)$$

where ν is a positive integer, and the dimensions of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ add to M. Our objective is to simplify (15) in order to obtain the necessary and sufficient conditions on $g_\ell(z), \ell = 0, 1$ for perfect reconstruction. It can be shown that (see Appendix A)

$$\hat{C}^T\hat{C} = 2M\left\{I + (-1)^{m+1}\begin{pmatrix}\Gamma_0 & \Gamma_1\\ \Gamma_1 & -\Gamma_0\end{pmatrix}\right\} \tag{17}$$

where Γ_0 and Γ_1 are as follows:

$$\Gamma_0 = \begin{cases}\begin{pmatrix}0 & 0\\ 0 & J_{M-m_1}\end{pmatrix}, & m_1 \le M\\ \begin{pmatrix}-J_{m_1-M} & 0\\ 0 & 0\end{pmatrix}, & m_1 > M,\end{cases}$$
$$\Gamma_1 = \begin{cases}\begin{pmatrix}J_{m_1} & 0\\ 0 & 0\end{pmatrix}, & m_1 \le M\\ \begin{pmatrix}0 & 0\\ 0 & J_{2M-m_1}\end{pmatrix}, & m_1 > M.\end{cases} \tag{18}$$

Substituting (17) into (15) and simplifying, we obtain

$$\frac{P(z)}{2M} = \Psi_0(-z^2) + z^{-1}[\Psi_1(-z^2) + (-1)^{m+1}\Psi_2(-z^2)] \tag{19}$$

where

$$\begin{aligned}\Psi_0(z) &= \tilde{g}_1(z)\Gamma_1 g_0(z) - z^{-1}\tilde{g}_0(z)\Gamma_1 g_1(z)\\ \Psi_1(z) &= \tilde{g}_0(z)g_0(z) + \tilde{g}_1(z)g_1(z)\\ \Psi_2(z) &= \tilde{g}_0(z)\Gamma_0 g_0(z) - \tilde{g}_1(z)\Gamma_0 g_1(z).\end{aligned} \tag{20}$$

Using (11), it can be shown that both Ψ_0 and Ψ_2 are identically 0 (Appendix B). Consequently, $P(z)$ simplifies to

$$\begin{aligned}P(z) &= 2Mz^{-1}\Psi_1(-z^2)\\ &= 2Mz^{-1}[\tilde{g}_0(-z^2)g_0(-z^2) + \tilde{g}_1(-z^2)g_1(-z^2)].\end{aligned}$$

The above matrix $P(z)$ is diagonal and should be a delay (for PR). In terms of $G_k(z)$

$$\tilde{G}_k(z)G_k(z) + \tilde{G}_{M+k}G_{M+k}(z) = \frac{1}{2M}, \quad 0 \le k \le M-1. \tag{21}$$

The above equations are the necessary and sufficient conditions on the polyphase filters $G_k(z)$ for perfect reconstruction. These conditions are exactly the same as those in [15] (where the length of $H(z)$ is $N = 2mM$). However, since the polyphase filters $G_k(z)$ do not have the same length and are related as in (11), the resulting consequence is very different from that in [15], as elaborated in the next section.

Discussion of Different Modes of (21)

On account of the relationship between the polyphase components given in (11), the conditions given by (21) can be satisfied by the following four modes:

1) $G_k(z)$ and $G_{M+k}(z)$ are unrelated and have same length: This is the same forms as those in [15] (the case when $N = 2mM$).
2) $G_k(z)$ and $G_{M+k}(z)$ are unrelated and have lengths $m+1$ and m, respectively. By comparing the coefficient of the largest power of z^{-1} (in (21)), One can show that either the first or the last coefficient of $G_k(z)$ must be 0.
3) $\tilde{G}_{M+k}(z) = z^{-r}G_k(z)$: Here, (21) becomes $G_k(z)G_{M+k}(z) = (1/4M)z^{-s}$, which implies that both $G_k(z)$ and $G_{M+k}(z)$ are delays.
4) $G_k(z)$ and $G_{M+k}(z)$ are linear-phase functions of length $m+1$ and m, respectively. Here, (21) becomes $G_k^2(z) + z^{-1}G_{M+k}^2(z) = z^{-s}$. Since the lengths of the two functions differ by 1, then the only solution is

$$\begin{aligned}G_k(z) &= z^{-n_0}, \quad G_{M+k}(z) = 0; \quad \text{even } s\\ G_k(z) &= 0, \quad G_{M+k}(z) = z^{-n_1}; \quad \text{odd } s\end{aligned} \tag{22}$$

for some n_0 and n_1.

For each k in the range $0 \le k \le M-1$, (21) will be in one of the above modes. Given M and m_1, there are some redundancies in k because of the relation in (11). In other words, there is a unique set of k (size $<M$) for which (21) must be satisfied. The following section will identify which values of k are in this set.

Case 1—Even M and Even m_1:

$$\begin{aligned}&m_1 < M; \begin{cases}0 \le k \le \frac{m_1}{2} - 1, & \text{mode b}\\ m_1 \le k \le \frac{M - m_1}{2} + m_1 - 1, & \text{mode a}\end{cases}\\ &m_1 = M; 0 \le k \le \frac{M}{2} - 1, \quad \text{mode b}\\ &m_1 > M; \begin{cases}\frac{m_1}{2} \le k \le M - 1, & \text{mode b}\\ 0 \le k \le \frac{m_1 - M}{2} - 1, & \text{mode a}\end{cases}\end{aligned} \tag{23}$$

Case 2—Odd M and Even m_1:

$$\begin{aligned}&m_1 < M; \begin{cases}0 \le k \le \frac{m_1}{2} - 1, & \text{mode b}\\ m_1 \le k \le \frac{M - m_1 - 1}{2} + m_1 - 1, & \text{mode a}\\ k = \frac{M + m_1 - 1}{2}, & \text{mode c}\end{cases}\\ &m_1 = M; \begin{cases}0 \le k \le \frac{M-3}{2}, & \text{mode b}\\ k = M - 1, & \text{mode c}\end{cases}\\ &m_1 > M; \begin{cases}\frac{m_1}{2} \le k \le M - 1, & \text{mode b}\\ 0 \le k \le \frac{m_1 - M - 3}{2}, & \text{mode a}\\ k = \frac{m_1 - M - 1}{2}, & \text{mode c}\end{cases}\end{aligned} \tag{24}$$

Case 3—Even M and Odd m_1:

$$\begin{aligned}&m_1 < M - 1; \begin{cases}0 \le k \le \frac{m_1 - 3}{2}, & \text{mode b}\\ m_1 \le k \le \frac{M - m_1 - 1}{2} + m_1 - 1, & \text{mode a}\\ k = \frac{M + m_1 - 1}{2}, & \text{mode c}\\ k = \frac{m_1 - 1}{2}, & \text{mode d}\end{cases}\\ &m_1 = M - 1; \begin{cases}0 \le k \le \frac{M-4}{2}, & \text{mode b}\\ k = M - 1, & \text{mode c}\\ k = \frac{M}{2} - 1, & \text{mode d}\end{cases}\end{aligned}$$

$$m_1 = M+1; \begin{cases} 1 \le k \le \dfrac{M-2}{2}, & \text{mode b} \\ k = 0, & \text{mode c} \\ k = \dfrac{M}{2}, & \text{mode d} \end{cases}$$

$$m_1 > M+1; \begin{cases} m_1 - M \le k \le \dfrac{m_1-3}{2}, & \text{mode b} \\ 0 \le k \le \dfrac{m_1-M-3}{2}, & \text{mode a} \\ k = \dfrac{m_1-M-1}{2}, & \text{mode c} \\ k = \dfrac{m_1-1}{2}, & \text{mode d} \end{cases} \tag{25}$$

Case 4—Odd M and Odd m_1:

$$m_1 < M; \begin{cases} 0 \le k \le \dfrac{m_1-3}{2}, & \text{mode b} \\ m_1 \le k \le \dfrac{M-m_1}{2} + m_1 - 1, & \text{mode a} \\ k = \dfrac{m_1-1}{2}, & \text{mode d} \end{cases}$$

$$m_1 = M; \begin{cases} 0 \le k \le \dfrac{M-3}{2}, & \text{mode b} \\ k = \dfrac{M-1}{2}, & \text{mode d} \end{cases}$$

$$m_1 > M; \begin{cases} m_1 - M \le k \le \dfrac{m_1-3}{2}, & \text{mode b} \\ 0 \le k \le \dfrac{m_1-M-2}{2}, & \text{mode a} \\ k = \dfrac{m_1-1}{2}, & \text{mode d} \end{cases} \tag{26}$$

In summary, we have shown in this section that the PR property for cosine-modulated filter bank with arbitrary length is the same as the case when the length is an even multiple of M. The pairs of polyphase components $G_k(z)$ and $G_{M+k}(z)$ (of the prototype filter $H(z)$) satisfy the power-complementary properties in (21). Since the prototype filter $H(z)$ has arbitrary length, $G_k(z)$ do not have the same length (unless $N = 2mM$). Depending on the lengths of $G_k(z)$ and $G_{M+k}(z)$, the power-complementary requirement imposes certain degenerate form on the polyphase components. These degenerate cases are summarized in modes c and d.

Recall from [15] that the polyphase transfer matrix of a paraunitary cosine-modulated filter bank is a parallel bank of power-complementary, two-channel filter bank cascaded by the modulation matrix $\hat{\boldsymbol{C}}$, as shown in Fig. 3(a) [15]. Fig. 3(b) shows the lattice structure for the two-channel lossless lattice [15], [29]. Since the filter bank considered in this paper is a general case of the one presented in [15], there must be a lattice structure for the case where the length is arbitrary. Section IV will elaborate on this lattice structure.

IV. Lattice Structures for Arbitrary-Length Cosine-Modulated Filter Banks

In this section, we describe a lattice implementation for the different modes described in Section III. The main motivation is to obtain a structure that ensures the PR property. In [15], a lattice structure for the cosine-modulated filter bank with prototype length $N = 2mM$ is given. This structure is based on the two-channel lossless lattice (shown in Fig. 3(b))

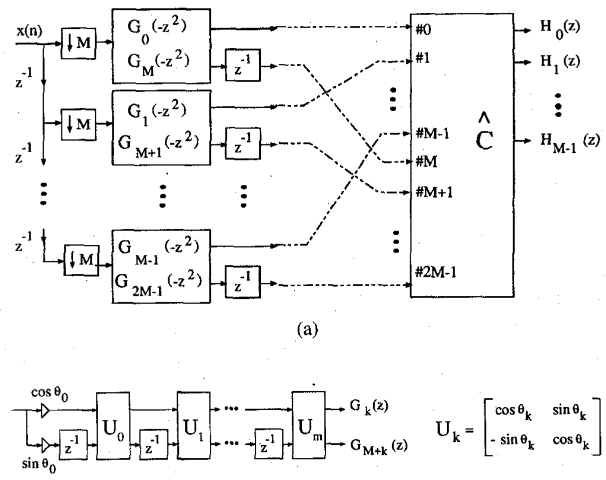


Fig. 3. (a) Lattice structure of the cosine-modulated filter bank; (b) two-channel lossless lattice.

introduced in [29], which ensures the power-complementary (PC) property between pairs of polyphase components. For the case considered in [15], all pairs of polyphase components have the same length and fall under *mode a*, as mentioned earlier. In this section, it is shown that each of the different modes can be implemented with a similar two-channel lattice structure but with different constraints on the angles.

Lattice Description

The two-channel lossless lattice in Fig. 3(b) can be described as follows:

$$\begin{bmatrix} G_k(z) \\ G_{M+k}(z) \end{bmatrix} = \boldsymbol{U}_m \Lambda(z) \boldsymbol{U}_{m-1} \Lambda(z) \cdots \boldsymbol{U}_1 \Lambda(z) \begin{bmatrix} c_0 \\ s_0 \end{bmatrix}$$

where

$$\boldsymbol{U}_k = \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ \sin\theta_k & -\cos\theta_k \end{bmatrix}, \qquad k = 1, 2, \cdots q \quad \text{and}$$

$$\Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}$$

$c_0 = \cos\theta_0$, and $s_0 = \sin\theta_0$. The main property of the two-channel lossless lattice is that for all values of θ_k, the polynomials $P(z)$ and $Q(z)$ satisfy the PC property.

Lattice for Mode a: In this mode, the polynomials $G_k(z)$ and $G_{M+k}(z)$ have the same length and are unrelated. There are no restrictions on any $\theta_k, k = 1, 2, \cdots (m-1)$, i.e., they can take any value in the range $[0, 2\pi]$. The polynomial pair $\{G_k(z), G_{M+k}(z)\}$ can be expressed as

$$\begin{bmatrix} G_k(z) \\ G_{M+k}(z) \end{bmatrix} = \boldsymbol{U}_{m-1} \Lambda(z) \boldsymbol{U}_{m-2} \Lambda(z) \cdots \boldsymbol{U}_1 \Lambda(z) \begin{bmatrix} c_0 \\ s_0 \end{bmatrix}.$$

Lattice for Mode b: In this mode, the polynomials $G_k(z)$ and $G_{M+k}(z)$ are unrelated but have lengths $m+1$ and m, respectively, and can be expressed as

$$\begin{bmatrix} G_k(z) \\ G_{M+k}(z) \end{bmatrix} = \boldsymbol{U}_m \Lambda(z) \boldsymbol{U}_{m-1} \Lambda(z) \cdots \boldsymbol{U}_1 \Lambda(z) \begin{bmatrix} c_0 \\ s_0 \end{bmatrix}.$$

A sufficient condition to satisfy this requirement is to let $\theta_k, k = 1, 2, \cdots (m-1)$ be arbitrary and force $\theta_m = \pi/2$. This can be readily verified as follows (where the superscript denotes the lattice stage number):

$$\begin{bmatrix} G_k(z) \\ G_{M+k}(z) \end{bmatrix} = \boldsymbol{U}_m \Lambda(z) \begin{bmatrix} G_k^{(m-1)}(z) \\ G_{M+k}^{(m-1)}(z) \end{bmatrix}.$$

With $\theta_m = \pi/2$, it can be seen that the length of $G_k(z)$ is $(m+1)$, whereas the length of $G_{M+k}(z)$ is m. It can also be seen that the first coefficient of $G_k(z)$ is zero.

Lattice for Mode c: In this mode, the polynomials $G_k(z)$ and $G_{M+k}(z)$ are both delays. A sufficient condition to satisfy this property is that all $\theta_k = 0$, or $\pi/2$, for all but one value of k, say k_1, and θ_{k_1} is arbitrary, i.e., it can take on any value in $[0, 2\pi]$. For example, the following is a possible case:

$$\theta_0 = \pi/4, \quad \text{and} \quad \theta_k = 0, \text{or } \pi/2, \qquad k = 1, 2, \cdots m.$$

Lattice for Mode d: In this mode, one of the polynomials $G_k(z), G_{M+k}(z)$ is a delay, and the other is zero. A sufficient condition to satisfy this condition is

$$\theta_k = 0, \quad \text{or } \pi/2, \qquad k = 0, 1, \cdots m.$$

As an example, consider a four-channel paraunitary cosine-modulated filter bank with length 34. This is case 1, which is summarized in (23). The PR conditions are

$$\tilde{G}_0(z)G_0(z) + \tilde{G}_4(z)G_4(z) = \frac{1}{8}, \quad \text{mode b}$$
$$\tilde{G}_2(z)G_2(z) + \tilde{G}_6(z)G_6(z) = \frac{1}{8}, \quad \text{mode a.}$$

The lossless lattice corresponding to $G_0(z)$ and $G_4(z)$ has five rotation angles $\theta_1, \cdots, \theta_5$. Since they are in mode b, therefore, the first four angles can take arbitrary values and the last angle $\theta_5 = \pi/2$. On the other hand, the lossless lattice corresponding to $G_2(z)$ and $G_6(z)$ has four rotation angles $\phi_1, \cdots, \phi_4$. Since they are in mode a, therefore, all four angles ϕ_k can take arbitrary values.

V. Quadratic-Constrained Least-Squares Formulation

The objective here is to design the lowpass prototype filter $H(z)$ such that its $2M$ polyphase components $G_k(z)$ satisfy the PR conditions in (21). Since the PR filter bank (where $N = 2mM$ [15]) can be parametrized using the lattice coefficients, a possible approach is to optimize these lattice coefficients to minimize the stopband error Υ of $H(e^{j\omega})$ (only the stopband of $H(e^{j\omega})$ is needed in the objective function since the filter bank satisfies the PC properties [10]). The drawback of the lattice approach is that the cost function Υ is a highly nonlinear function with respect to the lattice coefficients [15]. Consequently, PR filter banks, where the analysis filters have high stopband attenuation, are difficult to obtain. Instead of optimizing in the lattice coefficient space, we propose a design formulation that uses the filter coefficients directly [17], [25]. In other words, the cost function Υ and the PR conditions (21) can be expressed as quadratic functions of the filter coefficients. Consequently, as demonstrated in the examples below, PR filter banks with high stopband attenuation can be easily obtained.

Let $\boldsymbol{h}$ be a vector consisting of the first half (because of the linear-phase symmetry) of $h(n)$, i.e.

$$\boldsymbol{h} = \left(h(0)\ h(1)\ \cdots\ h\left(mM + \left\lfloor \frac{m_1+1}{2} \right\rfloor - 1\right)\right)^t. \tag{27}$$

Given M, m_1, and m, the appropriate PR conditions on the polyphase filters $G_k(z)$ can be found using (21) and (23)–(26). These PR conditions can be written in the form of

$$\begin{aligned} \boldsymbol{h}^t\boldsymbol{Q}_k\boldsymbol{h} &= 0, \\ \boldsymbol{h}^t\boldsymbol{S}_k\boldsymbol{h} &= 1 \end{aligned} \tag{28}$$

where $\boldsymbol{Q}_k$ and $\boldsymbol{S}_k$ depend on the filter bank parameters. We will elaborate on the forms of $\boldsymbol{Q}_k$ and $\boldsymbol{S}_k$ in the example sections below. Using the eigenfilter formulation [25], [27], the stopband error of $H(e^{j\omega})$ can be expressed in the quadratic form

$$\Upsilon = \boldsymbol{h}^t\boldsymbol{P}\boldsymbol{h} \tag{29}$$

where $\boldsymbol{P}$ is a real, symmetric, and positive-definite matrix, depending on the stopband cut-off frequency and N. Combining (28) and (29), the optimized filter $H(z)$ is precisely $\boldsymbol{h}_{\text{opt}}$ such that

$$\boldsymbol{h}_{\text{opt}} = \min_{\boldsymbol{h}} \boldsymbol{h}^t\boldsymbol{P}\boldsymbol{h} \quad \text{subject to} \quad \begin{cases} \boldsymbol{h}^t\boldsymbol{Q}_k\boldsymbol{h} = 0, \\ \boldsymbol{h}^t\boldsymbol{S}_k\boldsymbol{h} = 1. \end{cases} \tag{30}$$

In summary, we would like to formulate the design problem into a least squares optimization problem with quadratic constraints as in (30). Since $\boldsymbol{Q}_k$ is normally not positive definite, it is difficult to solve the above minimization problem. However, there are optimization procedures that approximately solve (30) by linearizing the quadratic constraints [28]. Using these procedures will yield an approximate solution (i.e., the constraints are not satisfied exactly). However, the errors are very small (in the order of 1×10^{-12}) and can be ignored in most practical cases. Having found the near-PR solution, we can synthesize the lattice structure to obtain the PR solution. Since the reconstruction error is very small, the resulting PR solution is very close to that of the near-PR one (the different in stopband attenuation between the two solutions is about 1 dB.)

The form of $\boldsymbol{Q}_k$ and $\boldsymbol{S}_k$ are derived in detail for the following two cases:

$$\begin{aligned} &\text{Case 1:} \quad M = 16, m = 8, m_1 = 0, N = 256 \\ &\text{Case 3:} \quad M = 8, m = 5, m_1 = 15, N = 95. \end{aligned} \tag{31}$$

(Other cases can be obtained similarly.)

Example 1—Case 1 (Even M and m_1) $m_1 = 0$: Since $m_1 < M$, the polyphase filters $G_k(z)$ must satisfy (23)

$$\tilde{G}_k(z)G_k(z) + \tilde{G}_{M+k}G_{M+k}(z) = \frac{1}{2M}, \quad 0 \le k \le \frac{M}{2} - 1, \quad \text{mode a.} \tag{32}$$

In terms of the variables $h(n)$, the polyphase filter $G_k(z)$ is

$$G_k(z) = \boldsymbol{h}^t \boldsymbol{V}_k \boldsymbol{e} \quad \text{and} \quad G_k(z^{-1}) = z^{m-1}\boldsymbol{h}^t \boldsymbol{V}_k \boldsymbol{J}\boldsymbol{e} \tag{33}$$

where

$$\begin{aligned} \boldsymbol{h} &= (h(0) \quad h(1) \quad \cdots \quad h(mM-1))^t \\ \boldsymbol{e} &= (1 \quad z^{-1} \quad \cdots \quad z^{-(m-1)})^t \\ [\boldsymbol{V}_k]_{i,j} &= \begin{cases} 1, & i = k + 2jM \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{34}$$

Note that the dimensions of $\boldsymbol{h}, \boldsymbol{e}$, and $\boldsymbol{V}_k$ are $(mM \times 1), (m \times 1)$, and $(mM \times m)$, respectively. Using the above relations on $G_k(z)$, (32) is simplified to

$$\boldsymbol{h}^t[\boldsymbol{V}_k \boldsymbol{J}\boldsymbol{e}\boldsymbol{e}^t\boldsymbol{V}_k^t + \boldsymbol{V}_{M+k}\boldsymbol{J}\boldsymbol{e}\boldsymbol{e}^t\boldsymbol{V}_{M+k}^t]\boldsymbol{h} = \frac{1}{2M}z^{-(m-1)}. \tag{35}$$

Substituting

$$\boldsymbol{e}\boldsymbol{e}^t = \sum_{n=0}^{2m-2} z^{-n}\boldsymbol{D}_n \quad \text{where} \quad [\boldsymbol{D}_n]_{i,j} = \begin{cases} 1, & i+j=n \\ 0, & \text{otherwise} \end{cases} \tag{36}$$

into (35) and simplifying, one obtains the following conditions on $\boldsymbol{h}$:

$$\begin{aligned} &\boldsymbol{h}^t[\boldsymbol{V}_k \boldsymbol{J}\boldsymbol{D}_n\boldsymbol{V}_k^t + \boldsymbol{V}_{M+k}\boldsymbol{J}\boldsymbol{D}_n\boldsymbol{V}_{M+k}^t]\boldsymbol{h} \\ &\quad = \begin{cases} 0, & 0 \le n \le m-2, \\ \dfrac{1}{2M}, & n = m-1 \end{cases} \end{aligned} \tag{37}$$

for k in the range $0 \le k \le M/2 - 1$. Note that the index n only goes to $m-1$ since (32) is symmetric. In summary, the $M/2$ PR conditions in (32) are rewritten as $mM/2$ quadratic constraints in $\boldsymbol{h}$ as in (37).

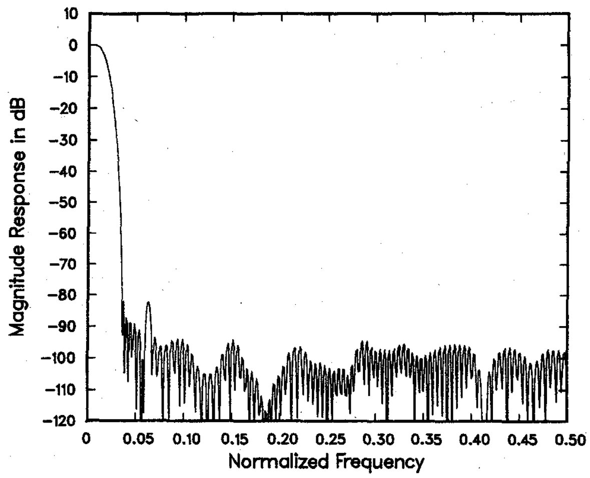
Design Procedure:

- Given M, m and the stopband edge of $H(z)$, compute $\boldsymbol{P}$ in (29) using the eigenfilter technique in [27].
- For each k in the range $0 \le k \le M/2 - 1$, compute the m conditions using (37). The total number of conditions is $mM/2$.
- Design a lowpass filter with the same specifications as in $H(z)$ and use its coefficients as an initialized value for $\boldsymbol{h}$ in the quadratic-constrained minimization problem (30). Use an IMSL subroutine [29] to solve the above minimization problem.

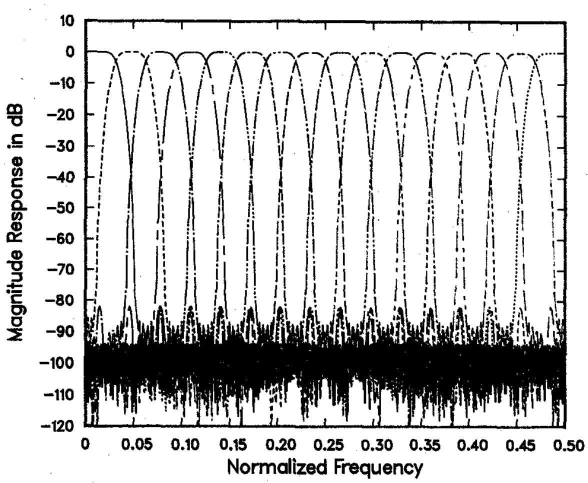
Let $M = 16, m = 8$, and $m_1 = 0$ (thus, $N = 256$). The magnitude responses of the optimized prototype filter $H(z)$ and analysis filters $H_k(z)$ are plotted in Fig. 4(a) and (b), respectively. Their stopband attenuation is about 82 dB.

Example 2—Case 3 (Even) M and Odd m_1) $m_1 = 2M-1$: Since $m_1 > M$, the polyphase filters $G_k(z)$ must satisfy (25)

$$\begin{aligned} &\tilde{G}_k(z)G_k(z) + \tilde{G}_{M+k}G_{M+k}(z) \\ &\quad = \frac{1}{2M}, \begin{cases} 0 \le k \le \dfrac{M}{2} - 2, & \text{mode a} \\ k = \dfrac{M}{2} - 1, & \text{mode c} \\ k = M-1, & \text{mode d.} \end{cases} \end{aligned} \tag{38}$$



(a)



(b)

Fig. 4. (a) Example 1: $M = 16, N = 256$. Magnitude response plot for the prototype filter $H(z)$; (b) $M = 16, N = 256$. Magnitude response plots for the analysis filters $H_k(z)$.

Note that the lengths of $G_k(z)$ are $m+1$ for k in the range of $0 \le k \le 2M-2$ and m for $k = 2M-1$. As pointed out in the discussion of various modes of (21), the conditions for modes c and d in (38) yield the following conditions on the polyphase filters $G_k(z)$:

$$\begin{aligned} &\tilde{G}_k(z)G_k(z) + \tilde{G}_{M+k}G_{M+k}(z) = \frac{1}{2M}, \\ &0 \le k \le \frac{M}{2} - 2 \\ &G_{2M-1} = \frac{1}{\sqrt{2M}}z^{-(m-1)/2}, \quad G_{M-1}(z) = 0, \\ &G_{(M/2)-1}(z) = \frac{1}{\sqrt{4M}}z^{-(m+1)/2}, \\ &G_{(3M/2)-1}(z) = \frac{1}{\sqrt{4M}}z^{-(m-1)/2}. \end{aligned} \tag{39}$$

In other words, some of the coefficients of $h(n)$ are restricted to values $0, 1/\sqrt{4M}$ and $1/\sqrt{2M}$, which in turn, limits the

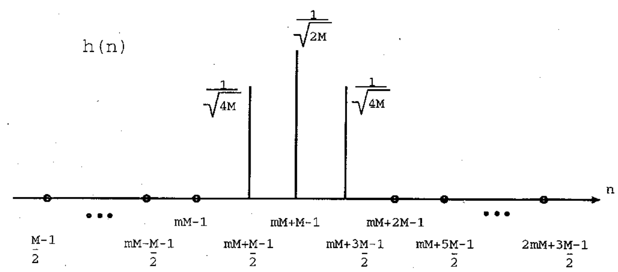


Fig. 5. Case 3. Even M and $m_1 = 2M - 1$. Constraints on $h(n)$.

performance of the filter $H(z)$. The indexes of the zero-coefficients are

$$\begin{aligned} G_{(M/2)-1}(z) &= \frac{1}{\sqrt{4M}} z^{-(m+1)/2} \Rightarrow \frac{m+1}{2} \text{ zeros at} \\ &\quad \frac{M}{2} - 1 + 2kM; \quad 0 \le k \le \frac{m-1}{2} \\ G_{M-1}(z) &= 0 \Rightarrow \frac{m+1}{2} \text{ zeros at} \\ &\quad M - 1 + 2kM; \quad 0 \le k \le \frac{m-1}{2} \\ G_{(3M/2)-1}(z) &= \frac{1}{\sqrt{4M}} z^{-(m-1)/2} \Rightarrow \frac{m-1}{2} \text{ zeros at} \\ &\quad \frac{3M}{2} - 1 + 2kM; \quad 0 \le k \le \frac{m-3}{2} \\ G_{2M-1}(z) &= \frac{1}{\sqrt{2M}} z^{-(m-1)/2} \Rightarrow \frac{m-3}{2} \text{ zeros at} \\ &\quad 2M - 1 + 2kM; \quad 0 \le k \le \frac{m-3}{2}. \end{aligned} \tag{40}$$

In other words, there are $4m$ zero-valued coefficients in $h(n)$ at the locations

$$h\left(k\frac{M}{2} - 1\right) = 0, \quad \begin{cases} 1 \le k \le 2m, \\ (2m+4) \le k \le (4m+3). \end{cases} \tag{41}$$

and

$$\begin{aligned} h\left(mM + \frac{M}{2} - 1\right) &= h\left(mM + \frac{3M}{2} - 1\right) = \frac{1}{\sqrt{4M}} \\ h(mM + M - 1) &= \frac{1}{\sqrt{2M}}. \end{aligned} \tag{42}$$

The conditions on (41) and (42) are depicted in Fig. 5. We observe that the mainlobe of $h(n)$ (from $n = mM - 1$ to $n = mM + 2M - 1$) spans $2M$ points, and the zero pattern is periodic with period $M/2$ outside the mainlobe region.

Recall that the filter $H(z)$ (Fig. 1(c)) is a lowpass filter with cutoff frequency at $\pi/2M$ (spectral factor of a $2M$th band filter). Thus, $H(z)$ should have a mainlobe of width $4M$ and a zero pattern with periodicity $2M$. The PR conditions yield the filter $H(z)$ with a mainlobe of width $2M$ and a zero pattern of period $M/2$. Consequently, the PR cosine-modulated filter bank with high stopband attenuation does not exist for case 3. With the above constraint on the filter coefficients $h(n)$, the design problem is formulated using the quadratic-constrained least-squares approach, which is similar to case 1 above. (We will skip the details here because of space limitations.)

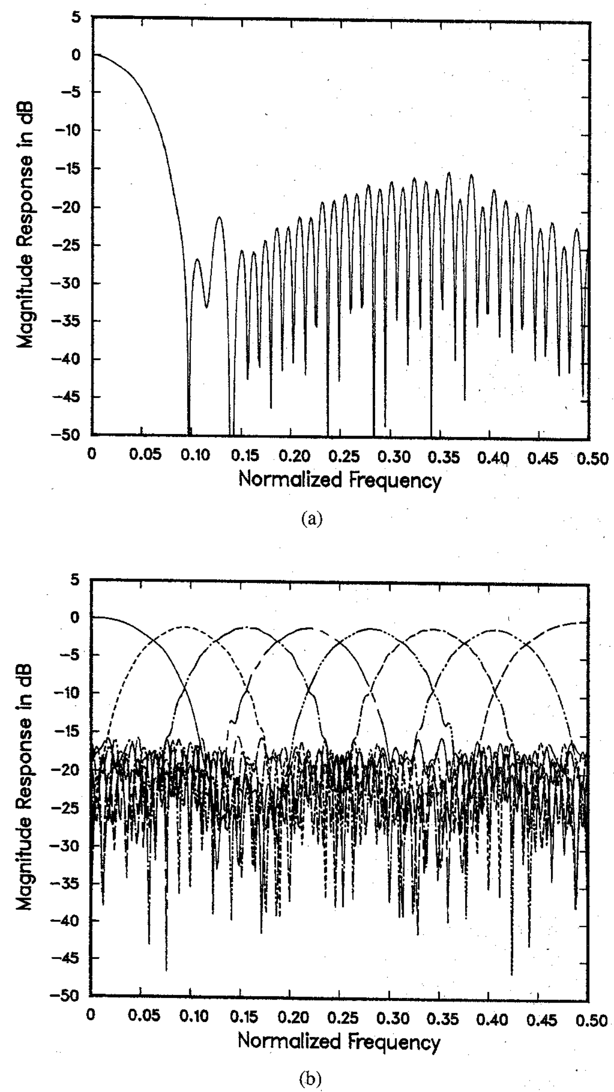


Fig. 6. (a) Example 2: $M = 8, N = 95$. Magnitude response plot for the prototype filter $H(z)$; (b) $M = 8, N = 95$. Magnitude response plots for the analysis filters $H_k(z)$.

As an example, an eight-channel PR filter bank with $m = 5$ and $m_1 = 15$ (thus, $N = 95$) is designed using the above method. The magnitude responses of the optimized prototype filter $H(z)$ and analysis filters $H_k(z)$ are plotted in Fig. 6(a) and (b), respectively. Their stopband attenuation is about 15 dB, which confirms the fact that high-stopband attenuation filter bank cannot be designed for this choice of M and m_1.

As demonstrated in the above two examples, designing a paraunitary cosine-modulated filter bank with arbitrary length can be tricky. The frequency responses of the resulting filters depend on the mode of the polyphase components. Since both modes c and d are very restrictive in the form of the polyphase components (they are either delays or 0), one might not be able to design a filter bank with high attenuation. Since both modes a and b allow arbitrary polyphase components, we believe that only the case of even M and even length (case 1; see (23)) is useful for obtaining high attenuation filter bank.

VI. The Compactly Supported Orthonormal Cosine-Modulated Wavelets

The wavelet transform [20]–[24] is a representation of a signal in terms of a set of basis functions that are obtained by dilations and shifts of a wavelet function. Thus, it provides a description of a signal at some level of resolution. It has been shown [24] that an Mth-band orthornormal wavelet can be obtained from an M-channel PR filter bank using an infinite tree structure. Moreover, the Mth-band wavelet would have L vanishing moments if and only if the function $H_0(e^{j\omega})$ has zeros of order L at frequencies $\omega_\ell = 2\ell\pi/M, 1 \le \ell \le M-1$ and $H_0(1) = \sqrt{M}$ [24]. In other words

$$\left|\frac{d^k}{d\omega^k} H_0(e^{j\omega})|_{\omega_\ell}\right|^2 = 0; \quad \begin{cases} 1 \le \ell \le M-1 \\ 0 \le k \le L-1. \end{cases} \tag{43}$$

where $\omega_\ell = 2\ell\pi/M$. The above conditions can be expressed in terms of the coefficient vector $\boldsymbol{h}$ as follows:

$$\left|\frac{d^k}{d\omega^k} H_0(e^{j\omega})|_{\omega_\ell}\right|^2 = \boldsymbol{h}^t(\boldsymbol{d}_{k\ell}\tilde{\boldsymbol{d}}_{k\ell})\boldsymbol{h} \tag{44}$$

where $[\boldsymbol{d}_{k\ell}]_i = 2(i)^k e^{-j\omega_\ell i}[\hat{C}]_{0,i} + 2(\hat{i})^k e^{-j\omega_\ell \hat{i}}[\hat{C}]_{0,\hat{i}}$, and $\hat{i} = N-1-i$. The condition $H_0(1) = \sqrt{M}$ can be written as $\boldsymbol{h}^t(\boldsymbol{d}_0\tilde{\boldsymbol{d}}_0)\boldsymbol{h} = \sqrt{M}$, where $[\boldsymbol{d}_0]_i = 2[\hat{C}]_{0,i} + 2[\hat{C}]_{0,N-1-i}$. The above $L(M-1)+1$ additional conditions (44) ensure that the wavelet has L vanishing moments. Together with the PR conditions in (21) (or (28)), the corresponding quadratic-constrained minimization problem is

$$\boldsymbol{h}_{\text{opt}} = \min_{\boldsymbol{h}} \Upsilon \quad \text{subject to} \quad \begin{cases} \boldsymbol{h}^t \boldsymbol{Q}_k \boldsymbol{h} = 0, \\ \boldsymbol{h}^t \boldsymbol{S}_k \boldsymbol{h} = 1 \\ \boldsymbol{h}^t(\boldsymbol{d}_0\tilde{\boldsymbol{d}}_0)\boldsymbol{h} = \sqrt{M} \\ \boldsymbol{h}^t(\boldsymbol{d}_{k\ell}\tilde{\boldsymbol{d}}_{k\ell})\boldsymbol{h} = 0. \end{cases} \tag{45}$$

Example 3: Let $M = 4, m = 6$, and $m_1 = 6$ (thus, $N = 54$; see case 1 (23)). Since $m_1 > M$, the polyphase filters $G_k(z)$ must satisfy

$$\tilde{G}_k(z)G_k(z) + \tilde{G}_{M+k}G_{M+k}(z) = \frac{1}{2M}; \quad \begin{cases} k = 3, & \text{mode b} \\ k = 0, & \text{mode a.} \end{cases} \tag{46}$$

For $k = 3$, the above condition is in mode b, which implies that the first coefficient of $G_3(z)$ (and, therefore, the last coefficient of $G_2(z)$) is 0. In other words, the impulse response $h(n)$ is zero at $n = 3$ and 50. The optimized analysis filters' responses (with no vanishing moments) are plotted in Fig. 7(a). With the above specifications, a compactly supported orthornormal cosine-modulated wavelet (with two vanishing moments) is designed using the above approach. The optimized wavelet functions $\Phi(\omega)$ and $\Psi_i(\omega)$ are plotted in Fig. 7(b).

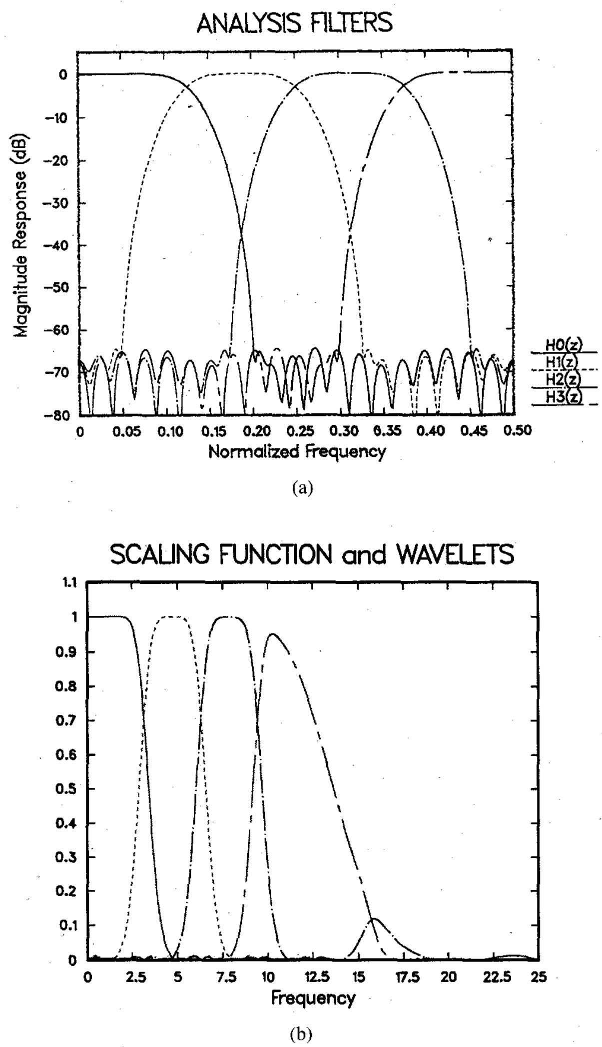


Fig. 7. (a) Example 3: $M = 4, N = 54, L = 0$. Magnitude response plots for the analysis filters $H_k(z)$; (b) $M = 4, N = 54, L = 2$. Scaling function and wavelets.

VII. Conclusion

The necessary and sufficient conditions for PR cosine-modulated filter bank designed from an arbitrary-length prototype filter are presented. The polyphase components $G_k(z)$ and $G_{M+k}(z)$ of the prototype filter form a two-channel losssless lattice. These lossless lattices are characterized by a set of rotation angles. The prototype filter design is formulated as a quadratic-constrained minimization problem. The optimized arbitrary-length PR cosine-modulated filter bank designed using this approach yields high stopband attenuation. It is shown that for certain choice of M and N, the design constraints (for PR) are too restrictive to yield a prototype filter with high attenuation. The quadratic-constrained minimization procedure is extended to design wavelets by incorporating the constraints to force several vanishing moments.

Appendix A

Given that $N = 2mM + m_1$ and

$$[\hat{C}]_{k,\ell} = 2\cos\left((2k+1)\frac{\pi}{2M}\left(\ell - \frac{N-1}{2}\right) + (-1)^k\frac{\pi}{4}\right)$$

we will show that the quantity $\hat{C}^T\hat{C}$ is as in (17). The (k,ℓ)th component of $\hat{C}^T\hat{C}$ is

$$\begin{aligned}[\hat{C}^T\hat{C}]_{k,\ell} &= \sum_{n=0}^{M-1}[\hat{C}^T]_{k,n}[\hat{C}]_{n,\ell}\\ &\doteq 4\sum_{n=0}^{M-1}\cos\left[(2n+1)\frac{\pi}{2M}\left(k-\frac{N-1}{2}\right)+\theta_n\right]\\ &\quad\cdot\cos\left[(2n+1)\frac{\pi}{2M}\left(\ell-\frac{N-1}{2}\right)+\theta_n\right]\\ &= 2\sum_{n=0}^{M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k+\ell-(N-1))\right.\\ &\quad\left.+2\theta_n\right]+2\sum_{n=0}^{M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k-\ell)\right]\\ &= [A]_{k,\ell}+[B]_{k,\ell}\end{aligned}$$

Calculation of $[B]_{k,\ell}$

$$\begin{aligned}[B]_{k,\ell} &= 2\sum_{n=0}^{M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k-\ell)\right]\\ &= \sum_{n=0}^{M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k-\ell)\right]+(-1)^{(k-\ell)}\\ &\quad\cdot\sum_{n=M}^{2M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k-\ell)\right].\end{aligned}$$

- If $(k-\ell)$ is even, then

$$\begin{aligned}[B]_{k,\ell} &= \sum_{n=0}^{2M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k-\ell)\right]\\ &= \begin{cases}0; & k\neq\ell,\\ 2M; & k=\ell.\end{cases}\end{aligned}$$

- If $(k-\ell)$ is odd, then

$$\begin{aligned}[B]_{k,\ell} &= \sum_{n=0}^{M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k-\ell)\right]\\ &\quad-\sum_{n=M}^{2M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k-\ell)\right]\end{aligned}$$

which is 0 since for each index n in the first summation, there is a corresponding index $(2M-1-n)$ in the second summation that yields the same value. Thus

$$B = 2MI.$$

Calculation of $[A]_{k,\ell}$

$$\begin{aligned}[A]_{k,\ell} &= 2\sum_{n=0}^{M-1}\cos\left[(2n+1)\frac{\pi}{2M}(k+\ell-(N-1))+2\theta_n\right]\\ &= -2(-1)^m\sum_{n=0}^{M-1}(-1)^n\\ &\quad\cdot\sin\left[(2n+1)\frac{\pi}{2M}(k+\ell-m_1+1)\right].\end{aligned}$$

- If $(k+\ell-m_1+1)=M$, then $[A]_{k,\ell}=-2M(-1)^m$.
- If $(k+\ell-m_1+1)=-M$ or $3M$, then $[A]_{k,\ell}=2M(-1)^m$.
- Otherwise, $[A]_{k,\ell}=0$.

In summary, $[A]_{k,\ell}$ is

$$[A]_{k,\ell}=\begin{cases}-2M(-1)^m; & i+j-m_1+1=M\\ 2M(-1)^m; & i+j-m_1+1=-M,3M\\ 0; & \text{otherwise.}\end{cases}$$

A can be written in another way, i.e.

$$A = 2M(-1)^{m+1}\begin{pmatrix}\Gamma_0 & \Gamma_1\\ \Gamma_1 & -\Gamma_0\end{pmatrix}$$

where Γ_0 and Γ_1 are given in (18).

Appendix B

Calculation of $\Psi_0(z)=\tilde{g}_1(z)\Gamma_1 g_0(z)-z^{-1}\tilde{g}_0(z)\Gamma_1 g_1(z)$

Since the matrices $g_0(z)$ and $g_1(z)$ are diagonal matrices and Γ_1 is as in (18), therefore, $\Psi_0(z)$ also has the same form as in Γ_1. There are two cases: $m_1\le M$ and $m_1>M$.

- $m_1\le M$: Using the definitions of $g_0(z), g_1(z)$, and $\Gamma_1, \Psi_0(z)$ yields the following m_1 equations:

$$\tilde{G}_{M+k}(z)G_{m_1-1-k}(z)-z^{-1}\tilde{G}_k(z)G_{M+m_1-1-k}(z),\quad 0\le k\le m_1-1. \tag{B.1}$$

Since $H(z)$ is a linear-phase function, its polyphase functions satisfy (11), which in turn, simplify the above m_1 equations as follows:

$$\begin{aligned}&(z^{m-1}G_{M+m_1-1-k}(z))(G_{m_1-1-k}(z))\\ &\quad-z^{-1}(z^mG_{m_1-1-k}(z))(G_{M+m_1-1-k}(z))=0.\end{aligned} \tag{B.2}$$

- $m_1>M$: Using the definitions of $g_0(z), g_1(z)$, and $\Gamma_1, \Psi_0(z)$ yields the following $2M-m_1+1$ equations:

$$\tilde{G}_{m_1+k}(z)G_{M-1-k}(z)-z^{-1}\tilde{G}_{m_1-M+k}(z)G_{2M-1-k}(z),\quad 0\le k\le 2M-m_1. \tag{B.3}$$

Sustituting (11) into (B.3) and noticing that $\tilde{G}_{m_1+k}(z)$ and $G_{m_1-M+k}(z)$ have lengths m and $m+1$, respectively, we have

$$\begin{aligned}&(z^{m-1}G_{2M-k-1}(z))(G_{M-1-k}(z))\\ &\quad-z^{-1}(z^mG_{M-1-k}(z))(G_{2M-1-k}(z))=0.\end{aligned} \tag{B.4}$$

Calculation of $\Psi_2(z) = \tilde{\boldsymbol{g}}_0(z)\Gamma_0\boldsymbol{g}_0(z) - \tilde{\boldsymbol{g}}_1(z)\Gamma_0\boldsymbol{g}_1(z)$

- $m_1 \leq M$: Using the definitions of $\boldsymbol{g}_0(z), \boldsymbol{g}_1(z)$, and $\Gamma_0, \Psi_2(z)$ yields the following $M - m_1$ equations:

$$\tilde{G}_{m_1+k}(z)G_{M-1-k}(z) - \tilde{G}_{M+m_1+k}(z)G_{2M-1-k}(z), \quad 0 \leq k \leq M - m_1 - 1. \tag{B.5}$$

Substituting (11) for $G_{m_1+k}(z)$ and $\tilde{G}_{M+m_1+k}(z)$ (both have lengths m), we have $\Psi_2(z) = 0$.

- $m_1 > M$: Similarly, $\Psi_2(z)$ becomes

$$-\tilde{G}_k(z)G_{m_1-M-1-k}(z) + \tilde{G}_{M+k}(z)G_{m_1-1-k}(z), \quad 0 \leq m_1 - M - 1. \tag{B.6}$$

which becomes 0 after using the relations in (11) (both $\tilde{G}_k(z)$ and $\tilde{G}_{M+k}(z)$ have lengths $m + 1$). In summary, as long as $H(z)$ is a linear-phase function, then both $\Psi_0(z)$ and $\Psi_2(z)$ are 0.

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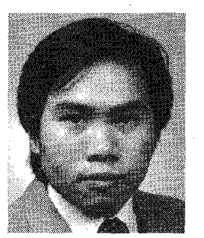

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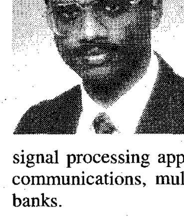

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