Space-time duality in multiple antenna channels

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Abstract

The concept of information transmission in a multiple antenna channel with scattering objects is studied from first physical principles. The amount of information that can be transported by electromagnetic radiation is related to the space-wavenumber and the time-frequency spectra of the system composed by the transmitting antennas and the scattering objects. The spatial information content is quantified in a similar fashion than its temporal counterpart, by reducing the inverse problem of field reconstruction to a communication problem in space, and determining the relevant communication modes of the channel by rigorously applying the sampling theorem on the field’s vector space. One consequence for narrow-band frequency transmission is that space and time can be decoupled, leading to a space-time information duality principle in the computation of the capacity of the radiating system. Interestingly, in the case of wide-band frequency transmission, it is shown that time and space cannot be decoupled and they jointly characterize the wave’s information content.

I. INTRODUCTION

The basic consideration that is the leitmotiv of this paper can be stated as follows: in propagation of electromagnetic (EM) waves, space and frequency are separate but intimately linked objects that pose fundamental limits on the amount of information carried by scattered fields. Resolving the amount of information that can be communicated through EM radiation is a venerable subject that has been treated by a great number of authors in different fields, often leading to rediscovering of basic facts time and again. Our attempt here is to present a unified approach based on the physical notions of space-wavenumber and time-frequency spectra of the propagating field.

We consider information sent in the form of waves, hence as a physical process: waves propagate in space through line of sight, reflection, scattering, and diffraction. Like any other physical phenomenon, this is governed by the laws of nature. These laws determine not only the process itself, but also the

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amount of diversity that a wave carries along its path. One kind of diversity is due to the wave being a function of time, hence being characterized by a frequency spectrum. Another kind of diversity is due to the wave propagating through space. In this case the wave interacts with the propagation environment and this interaction introduces spatial diversity that can be characterized in terms of spatial bandwidth. Both kinds of diversities influence the amount of information that can be communicated through wave propagation. The number of channels available for communication and the amount of information that can be transported over these channels depend on both the spatial and the frequency bandwidth of the radiating system.

This paper starts by considering the spatial bandwidth of a radiating system and its information content, and then considers the interaction between spatial and frequency bandwidths in terms of information capacity. The broad conclusions that can be drawn from our analysis are as follows. Space can be viewed as a capacity bearing object: the amount of information communicated by a scattering system increases with its size and with the size of the receiving antenna array. This is shown rigorously from physics, by applying the sampling theorem in the space domain. Furthermore, it turns out that in the narrow-band frequency case, time and space can be decoupled, and the information rate measured in bits per unit time and summed over all spatial information channels available for transmission, is equivalent to measuring information rate in bits per unit space and summing over all frequency information channels available for transmission, thus establishing a space-time information duality principle. On the other hand, in the case of wide-band frequency transmission, time and space are strongly coupled and they jointly characterize the information content of the radiating system.

Next, we wish to spend few words on the organization of the paper. Our analysis starts with a preliminary section where we identify the physical diversity limits of (frequency narrow-band) scattered fields, that were first introduced by Bucci and Franceschetti in the late nineteen eighties [2], [3], using a functional analysis approach. We then cast these results in a communication theoretic setting, viewing the inverse problem of field reconstruction through sampling interpolation as a communication problem. This provides the link between the notion of degrees of freedom of scattered fields and the corresponding information theoretic definition. That is, between the minimum number of independent parameters sufficient to completely reconstruct the radiated EM field up to arbitrary precision $\epsilon$, and the number of eigenvalues of the MIMO operator (scattering matrix) that are above an arbitrary level $\epsilon$. This link was recently pointed out by Migliore [11]. Following these preliminaries, in Section 3 we present a geometric corollary to Bucci and Franceschetti’s work to determine the relationship between the minimum non-redundant antenna spacing at the receiver, the wavelength, and the angular aperture of
the radiating system and we relate this corollary to the works of Poon et. al. [15] and Miller [13]. In Section IV, we examine the spatial information content of the scattered field in more detail, deriving a purely spatial additive white gaussian noise (AWGN) channel and its corresponding space-time version, and we point out an information duality principle between space and time, arising from the analysis of such channels. In Section V we extend the treatment to frequency wide-band transmission, showing the mutual interaction of space and time in the resolution of the information content of the propagating wave. Finally, in Section VI, we draw conclusions and outline directions for future work. Throughout the paper, we point out the practical implications of our analysis and we provide few illustrative examples.

II. PRELIMINARIES

A. Physics background: Bucci-Franceschetti diversity limits

Any real measurement system is invariably affected by a degree of uncertainty. This can stem from several factors, e.g., background noise, sensitivity and precision of the measuring system, dynamics of the measurement apparatus, and errors due to numerical approximation. Hence, it is both reasonable and necessary to consider two electromagnetic field vectors $E_1$ and $E_2$ indistinguishable if their difference, in a given norm, is below resolution $\epsilon$, that is

$$||E_1 - E_2|| < \epsilon.$$  (1)

Let us now consider a scattering system consisting of an arbitrary number of transmitters (sources) and scatterers, which are enclosed within a ball $B$ of finite radius $a$. The receivers are located on an observation domain lying on an analytical manifold $M$, which is assumed to be some wavelengths away from $B$. For the case of a one-dimensional (open) observation domain, $M$ is an arc spanning the range $[-S, S]$ in an arbitrary curvilinear coordinate system $s$. The arclength is normalized with respect to $r_m$, which is the minimum distance of $M$ from the center of $B$. All results described here are valid for one-dimensional observation domains and two-dimensional radiating system, but can be easily extended to higher dimensions. A schematic diagram of the scattering model is depicted in Figure 1. The primary sources are assumed to be uniformly bounded, so that the current density (including the polarization currents) induced on the scatterers satisfies the constraint

$$\int_B |J(r')| dr' \leq \eta,$$  (2)

where $\eta$ is a constant. In this section we shall consider narrowband sources, i.e., $J(r')$ consists of two impulses in the (temporal) spectral domain at angular frequencies $\omega$ and $-\omega$. 

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The field induced by the scattering system \( B \) on the observation manifold \( M \) can be expressed as

\[
E_0(r) = \int_B G_0(r, r') \cdot J(r') \, dr',
\]

where \( G_0 \) is the free-space dyadic Green’s function (cf., e.g., [6]),

\[
G_0(r, r') = \frac{j \omega \mu}{4\pi} \left( I + \frac{\nabla \nabla}{\beta^2} \right) \cdot \frac{\exp(-j\beta R)}{R},
\]

\( I \) being the identity matrix; \( \beta \) is the wavenumber satisfying \( \beta = \omega \sqrt{\varepsilon \mu} = \frac{\omega c}{c} \), where \( \varepsilon \) and \( \mu \) are the permeability and the permittivity of the medium, and \( c \) is the velocity of propagation of the wave; and \( R = \sqrt{(r - r') \cdot (r - r')} \). It is convenient at this point to extract the propagation factor \( \exp(-j\beta r) \) from \( E_0(r) \), and consider the reduced radiated field

\[
E(r) = E_0(r) \exp(j\beta r).
\]

Notice that the reduced field \( E \) is still represented by (3), provided that one substitutes \( G_0 \) with \( G = G_0 \exp(j\beta r) \), thus obtaining

\[
E(r) = \int_B G(r, r') \cdot J(r') \, dr'.
\]

In [2], [3], the following results were presented. It was shown that the reduced scattered field can be well approximated by space-bandlimited functions. The approximation error decreases with the bandwidth, undergoing a sharp transition in the vicinity of a critical value of bandwidth (see [3], Figure 3), which is henceforth referred to as the effective bandwidth \( W_0 \).

\[
W_0 = \beta a.
\]

The transition is very sharp for large scattering systems (i.e., for systems with \( a \gg \lambda \)). The reduced field can then be represented in terms of linear combinations of convenient basis elements. By appropriate choice of the basis, the coefficients of interpolation can be the sampled values of the field on the observation manifold [3]. Given a fixed precision level \( \epsilon \), the minimum number of basis functions necessary to obtain a satisfactory representation of the field elements is defined to be the degrees of freedom \( N_0 \). For large scattering systems, the number of degrees of freedom of the scattered fields can be identified with the Nyquist number

\[
N_0 = \frac{2SW_0}{\pi} = \frac{2S\beta a}{\pi},
\]

evaluated in terms of the effective bandwidth \( W_0 \). Note that \( 2S \) is the length of the detector surface \( M \), normalized with respect to \( r_m \), see Figure 1. For large scattering systems, the number of degrees of freedom is practically insensitive to the error level \( \epsilon \), i.e., a well-defined quantity [3]. For small scattering
systems, the dependence of the degrees of freedom on the error level $\epsilon$ was investigated numerically in [22], but this case is not considered here.

In the examples below, we consider some practical scenarios and illustrate sample values of the spatial bandwidth and of the number of degrees of freedom for various operating frequencies. Notice that the spatial bandwidth $W_0$ is normalized to the wavelength $\lambda$ and hence is a pure number. We shall come back to these examples several times throughout the paper.

**Example 2.1:** Consider a radiating sphere of radius $a = 1$ Km enclosing all the transmitters and the scatterers. Suppose the observation manifold is located at a distance of $r_m = 20$ Km, and the receiving antenna array is of (absolute) length $2S = 200$ m. This situation can correspond to a base station located on a hilltop serving mobiles located in a distant valley. In this case, $a/r_m = 0.05$, and $2S/r_m \approx 0.01$. In Table I, the spatial bandwidth ($W_0$) and the number of degrees of freedom ($N_0$) are computed for several carrier frequencies.

**Example 2.2:** Consider a radiating sphere of radius $a = 100$ m enclosing all the transmitters and scatterers. Suppose the observation manifold is a circle enclosing the radiating sphere, with radius $r_m = 250$ m. This is an idealized urban scenario, where a number of receiving antennas are placed around the mobiles and the scatterers. In this case, $a/r_m = 0.4$, and $2S/r_m = 2\pi$. In Table II, $W_0$ and $N_0$ are computed for several carrier frequencies.

**B. From physics to communication**

Let us now formally describe the communication problem following from the physics outlined in the previous section. This set up is analogous, with some minor variations, to the ones in Miller [13], Migliore [11], and Poon et. al. [15]. Let us consider the orthonormal basis expansion of the field measured over a finite spatial interval $[-S, S]$ on $\mathcal{M}$,

$$E(r) = \sum_{i=0}^{\infty} \alpha_i \Psi_i(r),$$

(9)

where $\{\alpha_i\}_{i=1}^{\infty}$ are the coefficients of expansion, and $\{\Psi_i(r)\}_{i=1}^{\infty}$ comprise a complete orthonormal (vector) basis set. An arbitrary source current density $J(r')$ inside $\mathcal{B}$ can also be described by a complete orthonormal (vector) basis set $\{\Phi_l(r')\}_{l=1}^{\infty}$ as

$$J(r') = \sum_{l=1}^{\infty} \beta_l \Phi_l(r'), \quad r' \in \mathcal{B},$$

(10)

where $\{\beta_l\}_{l=1}^{\infty}$ are the coefficients of expansion. Next, substituting (9) and (10) into (6), we have

$$\alpha_k = \sum_{l=1}^{\infty} h_{kl} \beta_l, \quad k = 1, 2, \ldots ,$$

(11)
where \( h_{kl} = \int_M \int_B \Psi_k^*(\mathbf{r}) \cdot \mathbf{G} (\mathbf{r}, \mathbf{r}') \cdot \Phi_l (\mathbf{r}') \, d\mathbf{r} d\mathbf{r}' \). The components \( h_{kl} \) can be thought as the coupling coefficients between the transmission mode \( l \) in the ball \( B \) and the reception mode \( k \) in the manifold \( M \).

In matrix form, (11) is equivalent to

\[
\alpha = H \cdot \beta,
\]

where \( \alpha = \{\alpha_k\}_{k=1}^{\infty} \) and \( \beta = \{\beta_l\}_{l=1}^{\infty} \) are the vectors of coefficients and the elements of the matrix \( H \) are the components \( \{h_{kl}\} \). The matrix \( H \) can be viewed as the communication operator (viz. the scattering matrix) between the radiating ball and the receiving manifold. The parameter \( R = \text{rank}(H) \) defines the number of degrees of freedom of the communication system, as it determines the number of parallel independent communication modes. For large scattering systems, recall that the number of basis functions \( \{\Psi_k\} \) required to represent the received field intensity within a given precision tends to \( N_0 \), the number of degrees of freedom of the scattered field. Therefore, the linear subspace spanning the rows of \( H \) is at most of dimension \( N_0 \), so that \( R \leq N_0 \). This is the main observation of [11]. Note that the sources and the scatterers also play an important role in the determination of the exact value of \( R \), since the rank of \( H \) is also dependent on the size of the linear subspace spanned by the columns of \( H \), which is dictated by the number of orthonormal basis functions \( \{\phi_l\} \) required to represent the source current density. In other words, the EM degrees of freedom represent an upper bound on the degrees of freedom of the MIMO communication channel obtained by considering all possible configurations of sources and scatterers that can occur inside the radiating ball.

Due to the compactness of the radiation operator, it is possible to perform an eigenvalue decomposition of the infinite dimensional matrix \( H \), where only a finite number of diagonal elements are essentially nonzero. The resulting eigenfunctions yield the communication modes. Formally, we can write

\[
H = U \cdot \Lambda \cdot V^*,
\]

where \( U \) and \( V \) are (infinite-dimensional) unitary matrices, and \( \Lambda \) is a diagonal matrix with \( R \) nonzero components \( \{\ell_1, \ldots, \ell_R\} \). Without loss of generality, we assume that \( \Lambda \) is normalized, i.e., \( \sum_{i=1}^{R} \ell_i^2 = 1 \). Then (12) is transformed into

\[
\alpha' = \Lambda \cdot \beta',
\]

where the components of the linear vectors \( \alpha' = U^* \cdot \alpha \) and \( \beta' = V^* \cdot \beta \) constitute the communication modes. This eigenvalue decomposition allows to decompose the continuous space MIMO channel into a finite number of orthogonal SISO subchannels. The components \( \{\ell_1, \ldots, \ell_R\} \) determine the coupling strengths of the respective subchannels, and are critical in the EM characterization of the capacity of the space-time MIMO channel, as we shall see in Section IV-B.
III. MINIMUM NON REDUNDANT ANTENNA SPACING

We now derive a geometric corollary to Bucci and Franceschetti’s work which provides an elegant interpretation of the number of degrees of freedom, and leads to the determination of the minimum non-redundant antennas spacing in terms of wavelength and angular aperture of the system. We explicitly note that this minimum spacing is not heuristic, but it follows directly from physics and from application of the sampling theorem in the space domain, in light of the considerations made in the previous section.

For the purpose of this discussion, we explicitly indicate the normalization factor $r_m$ and assume that the non-normalized length of the observation domain is $2S$. It follows from (8), if the manifold is at approximately constant distance from the radiating system, then

$$N_0 = \left(\frac{2S}{r_m}\right) \cdot \left(\frac{2a}{\lambda}\right) \approx \left(\frac{2a}{\lambda}\right) \cdot \Omega_r,$$

(15)

where $\Omega_r$ is the angle subtended by the receiving manifold on the transmitting volume (see Figure 2(a)). Thus, the number of degrees of freedom is approximately equal to the product of the angle subtended by the receiving manifold on the transmitting sphere times the size of the transmitter normalized with respect to the carrier wavelength. Similarly,

$$N_0 = \left(\frac{2a}{r_m}\right) \cdot \left(\frac{2S}{\lambda}\right) \approx \left(\frac{2S}{\lambda}\right) \cdot \Omega_t,$$

(16)

where $\Omega_t$ is the angle subtended by the transmitting volume on the receiving manifold (see Figure 2(b)). Thus, the number of degrees of freedom is approximately equal to the product of the angle subtended by the transmitting volume on the receiving manifold times the size of the receiver normalized with respect to the carrier wavelength.

Above formulas show that for fixed size of the radiating system, the number of degrees of freedom increases with the angular resolution of the receiver; while for fixed size of the receiver it increases with the angular spread of the transmitter. This interpretation recovers the results in [15]. These authors considered a slightly different, but essentially equivalent model in which clusters of scatterers are located in the far field with respect to both the sources and the detectors. The number of degrees of freedom for this model is reported to be the minimum of two product terms, each of which are of the form (15) and (16). Furthermore, in a previous work [13] again the same kind of formula appears. In fact, this size times angular resolution formula has been known, at least non-rigorously, since the nineteen fifties, dating back at least to the florentine school of Toraldo di Francia [19]–[21]. The formal connection between the number of degrees of freedom and the sampling rate, a direct consequence of Bucci and Franceschetti’s band-limitation property, constitutes the derivation provided above.
It has also been reported in [15] and [17] that, for linear transmitting and receiving antenna arrays, the minimum antenna spacing to obtain a non-redundant representation of the field at the receiver is such that the antenna elements are separated by a distance \( \lambda/2 \). We now show that this minimum spacing also depends on the angle subtended by the transmitting ball at the receiver. From (15) or (16) we have that

\[
\frac{2S}{N_0} \frac{a}{r_m} = \frac{\lambda}{2},
\]

(17)

Since \( 2S/N_0 \) is the Nyquist sampling interval of the space bandlimited field, it follows from (17) that the minimum non-redundant antenna spacing is given by

\[
\Delta s = \frac{2S}{N_0} = \frac{r_m \lambda}{2a}.
\]

(18)

Therefore, half a wavelength is the minimum non-redundant antenna spacing only when \( r_m \approx a \) and the manifold embraces the sphere, while in general for \( r_m \gg a \) the appropriate non-redundant spacing is given by (18). Notice that \( \Delta s \) increases with \( r_m \). The reason for this should be clear: consider for example different values of \( r_m \) corresponding to different concentric arcs subtended by the same angle \( \theta \) at the center of the ball \( B \). The lengths of these arcs are different: the one closest to the radiating ball is of the shortest (absolute) length. However, the normalized lengths with respect to their distance from the center of \( B \) are the same. Therefore, the number of samples needed for reconstruction in each of these sections are the same, which implies that the spacing between the adjacent samples is larger for the ones located at larger distance from the center. This agrees with practical design principles of wireless communication systems, where the antennas at the cellular base station are placed several wavelengths apart when the base station is far from the radiating system, as for example in a hilltop setting, while they are tightly packed together in microcell base stations, where the base station is close to the radiating system, as for example in an urban setting (cf. [5] and the following examples).

**Example 3.1:** Recall the hilltop example (cf. Example 2.1) with \( a = 1 \text{ Km} \), \( r_m = 20 \text{ Km} \), and \( 2S = 200 \text{ m} \). In this case, with \( f_c = 900 \text{ MHz} \) (i.e., \( \lambda = 0.33 \text{ m} \)), the minimum antenna spacing is given by \( \Delta s = \frac{2S}{N_0} = \frac{200}{60} = 3.33 \text{m} = 10 \lambda \). This can also be seen from (18) since \( \frac{\Delta s}{\lambda} = \frac{1}{10} = \frac{r_m}{2a} = 10 \). Hence, the antennas should be widely separated.

Consider now the urban example (cf. Example 2.2) with \( a = 100 \text{ m} \), \( r_m = 250 \text{ m} \), and \( 2S = 2\pi r_m \). In this case, with \( f_c = 900 \text{ MHz} \), the minimum antenna spacing is given by \( \Delta s = \frac{2S}{N_0} = \frac{2\pi \times 250}{300} = 0.417 \text{m} = 1.25 \lambda \). This is again consistent with (18), since \( \frac{\Delta s}{\lambda} = \frac{r_m}{20} = 1.25 \). Hence, in this case, the antennas can be tightly packed together.

Both of the cases above are in agreement with practical design principles of cellular systems, see [5].
IV. INFORMATION DUALITY PRINCIPLE

We are now ready to give an information theoretic characterization of the radiated field. This is the natural extension of the results in [11]. We first introduce a pure spatial and then a spatio-temporal MIMO communication channel model, based on interpolation through sampled field points; then we present a space-time information duality principle that arises from the analysis of these two channels.

A. A spatial AWGN communication channel

We start with a very simple model of additive white Gaussian noise (AWGN) spatial communication channel that is progressively refined in the next sections. The channel we consider here is a purely spatial channel, in the sense that information is conveyed from the radiating ball to the observation domain in a single time step, using the spatial communication modes. Furthermore, for the sake of simplicity, this channel is constructed at the receiver under the assumption that the source is capable of generating a given field pattern on the manifold \( \mathcal{M} \) (this assumption will be removed in the next sections). The codewords are the electric field expansion coefficients measured on the manifold, corresponding to a complete orthonormal basis function set \( \{ \psi_k \} \), and the background noise due to measurement errors and approximation errors is cumulatively modeled as independent Gaussian random variables that are added to each expansion coefficient.

Consider a codeword of \( n \) symbols \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) that we wish to communicate. We associate these symbols to the expansion coefficients of the field on the manifold \( \mathcal{M} \) and assume that the electric field \( E(\mathbf{r}) \) on the whole manifold can be recovered from these symbols within a given precision level by using non-redundant sampling interpolation in space. We first derive a bound on the average spatial power measured in the interval \((-S, S)\). This can be obtained from the constraint on the source current density (2) and the reduced field equation (6) as follows.

\[
|E(\mathbf{r})| = \left| \int_{B} G(\mathbf{r}, \mathbf{r}') \cdot J(\mathbf{r}') d\mathbf{r}' \right| \leq \sup_{\mathbf{r}'} |G(\mathbf{r}, \mathbf{r}')| \int_{B} |J(\mathbf{r}')| d\mathbf{r}' \leq \sup_{\mathbf{r}'} |G(\mathbf{r}, \mathbf{r}')| \eta, \quad (19)
\]

where the last inequality follows from (2). Squaring and integrating both sides of (19), we have

\[
\frac{1}{2S} \int_{\mathcal{M}} |E(\mathbf{r})|^2 d\mathbf{r} \leq \frac{\eta^2}{2S} \int_{\mathcal{M}} \left( \sup_{\mathbf{r}'} |G(\mathbf{r}, \mathbf{r}')| \right)^2 d\mathbf{r} = P, \quad (20)
\]

where \( P \) is a constant because the integrand is bounded, since the manifold \( \mathcal{M} \) is outside the ball \( B \) so that \( \mathbf{r} \neq \mathbf{r}' \); and the integral is over a bounded support \( 2S \). Notice that \( P \) is (but for a normalizing resistance) the average spatial power over the manifold. Inequality (20) clearly holds also for the co-polar component of the field, \( E(\mathbf{r}) \), which we consider next.
Assuming an orthonormal basis expansion of the field, by (20) it follows that we have the codeword constraint
\[ \frac{1}{n} \sum_{i=1}^{n} |\alpha_i|^2 = \frac{1}{n} \int_{-S}^{S} |E(r)|^2 dr \leq \frac{2SP}{n}. \]

Notice that $2S/n$ stays constant provided that the sampling rate over the manifold has been fixed and the spatial codeword length $S$ grows with $n$.

We now introduce the background noise due to measurement and approximation errors at the receiver. Let a Gaussian random variable $N(0, \sigma^2)$ be added independently to each transmitted symbol $\alpha_i$, so that for a transmitted codeword $(\alpha_1, \ldots, \alpha_n)$, the corresponding received codeword $(\gamma_1, \ldots, \gamma_n)$ is obtained by
\[ \gamma_i = \alpha_i + z_i, \quad i = 1, \ldots, n, \]
where the $z_i$’s are realizations of i.i.d. $N(0, \sigma^2)$ random variables. We remark here that the index $i$ is a space index. We can now use Shannon’s formula to state that,

**Theorem 1:** The information capacity of the spatial additive Gaussian noise channel described above and subject to the constraint (21) is given by
\[ C = \frac{1}{2} \log \left(1 + \frac{P}{\sigma^2}\right) \text{ bits per spatial symbol.} \]

It is appropriate at this point to summarize the assumptions we need for above capacity formula to hold. First of all, we need a non-redundant sampling representation of the field. This ensures that the coefficients in the codeword, being essentially the same number as the degrees of freedom $N_0$ of the field, are independent. Since the field is effectively bandlimited [2], this can be ensured by choosing appropriate basis functions. The second point to notice is that we need the source to be capable of generating the desired pattern of field coefficients $\{\alpha_i\}$ on the manifold $\mathcal{M}$. This might not be true in general and depends on the actual configuration of the sources and the scatterers inside the ball, which determines the channel communication modes. For example, a portion of the manifold might be shaded by absorbing obstacles present inside the ball. In this case, the coefficients $\{\alpha_i\}$ on this portion of the manifold cannot be excited. In fact, recall from Section II-B that the number of degrees of freedom of the field measured on the manifold $\mathcal{M}$ is an upper bound on the number of degrees of freedom of the communication system. This upper bound can be achieved considering all possible configurations of sources and scatterers inside the ball. In the next sections we shall make the dependence on the sources and the communication modes explicit. Finally, above capacity formula is only valid in the limit of large
codeword size, that is as $n \to \infty$. This requires the manifold size to diverge as well, so that $2SP/n$ remains constant. In practice, we are constrained to a finite number of observation points placed on a manifold of finite size. We shall address this last problem in detail at the end of this section.

We now perform one more step to obtain the capacity per unit normalized space over the channel. This follows by introducing the additional physical constraint of the effective spatial bandwidth $W_0$. This limits the amount of detectable variation over the manifold of any transmitted signal. By (8), at most $N_0 = 2SW_0/\pi$ different symbols can be transmitted over a spatial region of normalized length $2S$. Thus, the capacity per unit normalized space is given by

$$C = \frac{N_0}{2S} \log \left( 1 + \frac{P}{\sigma^2} \right) = \frac{W_0}{2\pi} \log \left( 1 + \frac{P}{\sigma^2} \right) \text{bits per normalized length.} \quad (24)$$

The noise is also related to the spatial bandwidth $W_0$. The average power of the (spatial) white Gaussian noise process is proportional to the spatial bandwidth, so that $\sigma^2 = N_s W_s$, where $W_s = \frac{W_0}{2\pi}$ and $N_s$ is the constant (spatial) noise power spectral density. Thus, we recover the spatial version of the Shannon-Hartley formula for communication over a bandlimited channel subject to additive white Gaussian noise.

**Theorem 2:** The information capacity of the spatial MIMO radiating system depicted in Figure 1, where transmitters are narrowband with wavelength $\lambda$, and the receiving array is subject to additive white Gaussian noise is given by

$$C = W_s \log \left( 1 + \frac{P}{N_s W_s} \right) \text{bits per normalized length,} \quad (25)$$

where $W_s = \frac{W_0}{2\pi} = \frac{\alpha}{\lambda}$.

Some observations are now in order. First of all, we note that spatial capacity is measured in bits per unit of (normalized) space, which correspond to the bits per unit time in the usual temporal formula. Furthermore, the spatial capacity is associated to an observation interval of (normalized) length $2S$; similarly, the temporal capacity is associated to a time interval of $T$ seconds. In fact, the number of bits that can be transmitted over the manifold scales with its (normalized) size. In the temporal analogy this corresponds to the trivial observation that the amount of information transmitted over a time interval $T' = 2T$ corresponds to twice the amount of information sent over a time interval $T$. In the spatial domain, the increase is due to a longer receiving array which enhances the angular resolution, i.e., the detection capability of the receiver. In the temporal domain this corresponds to transmitting more symbols over the channel using a longer time window. We also note that the spatial capacity scales with the spatial bandwidth $W_0 = \beta a$, i.e., with the (wavelength-normalized) size of the radiating system in a same fashion as the temporal capacity scales with the frequency bandwidth. Since $W_0$ is proportional to the size of the
radiating system, this means that a larger radiating system increases the information capacity by working as a lens enhancing its focusing capability. Furthermore, $W_0$ is inversely proportional to the wavelength, which means that transmitting at higher frequencies (i.e. shorter wavelengths) can lead, in principle, to higher information rates.

Next, we turn to the problem that in the usual temporal formulation the capacity formula is derived by letting $T \to \infty$, where $T$ is the time interval of transmission and reception of the continuous function of time representing the codeword symbols. Instead, in the spatial formulation the normalized spatial interval is bounded by the total angular spread ($2\pi$ in the case of a one-dimensional observation domain). This invariably implies bounded spatial codeword lengths, and hence that the capacity formulas derived above are not achievable, but rather provide upper bounds on the achievable information rates of the spatial channel under consideration. We emphasize that this is true even if we stretch the observation domain to infinity, since the number of degrees of freedom of the electric field induced on the (unbounded) observation manifold by sources and scatterers enclosed in a bounded volume is still finite, and bounded by the total angular spread, as it was discussed at the end of Section III.

In this context, it is therefore useful to consider the notion of error exponents arising because the normalized spatial interval is bounded. The probability of an error event, which occurs when the detected codeword is different from the transmitted codeword, decays exponentially with $n$, the number of symbols in the codeword. The exponent of the error probability indicates the rate of decay of the error event, and is a useful channel parameter. For the AWGN channel, upper bounds on the error exponent are well known (cf., e.g., [7], Theorem 7.4.4). In the following example we apply such bounds showing that, with a larger $N_0$, information rates closer to capacity can be achieved. A large value of $N_0$ can be achieved by using a higher carrier frequency, and/or by using a large scattering system, and/or by using a large observation domain size. It should be noted that these estimates are based on the upper bounds on error probability, and the actual achievable rates can in fact be higher.

**Example 4.1:** Recall the hilltop example (Example 2.1) with $a = 1 \text{ Km}$, $r_m = 20 \text{ Km}$, and $2S = 200 \text{ m}$, and the urban example (Example 2.2) with $a = 100 \text{ m}$, $r_m = 250 \text{ m}$, and $2S = 2\pi r_m$. In Table III, the ratio of the maximum achievable information rate to the channel capacity ($R_{\text{max}}/C$) corresponding to $P_e \leq 10^{-6}$ for SNR = 0 dB are computed for several carrier frequencies for all of these scenarios.

**B. A space-time AWGN communication channel**

In this section, the source geometry is taken into account so that an explicit relation between input (represented by the current density inside the ball) and output (represented by the observed EM field on
the manifold) is obtained. Furthermore, we consider transmission over both space and time. There has been a plethora of recent publications that provide a space-time MIMO model based on the physics of propagation [1], [8]–[13], [15]–[17], [23]. The common denominator of all of these works follows the outline given in Section II-B, where the MIMO channel is decomposed into a set of $\mathcal{R}$ parallel SISO subchannels, as given by equation (14).

As in the case of the purely spatial channel model, we can introduce the effects of background noise via a simple AWGN model. From (14) it follows that the measured communication modes are given by

$$\gamma_i' = \alpha_i' + z_i = \ell_i \beta_i' + z_i, \quad i = 1, \cdots, \mathcal{R},$$

where $\{z_i\}$ are realizations of i.i.d. $\mathcal{N}(0, \sigma^2)$ random variables which are added to each space component at a given instant of time; $\{\ell_i, i = 1, \cdots, \mathcal{R}\}$ are the nonzero diagonal elements of $\Lambda$; and the input coefficients are subject to a power constraint

$$\frac{1}{\mathcal{R}} \sum_{i=1}^{\mathcal{R}} |\beta_i'|^2 \leq P',$$

where $P'$ can be immediately related to $\eta$. Recall from Section II-B that $\mathcal{R}$ is the number of independent parallel SISO subchannels, as determined by the singular value decomposition of the channel matrix $H$; furthermore, $\mathcal{R} \leq N_0$, where $N_0$ is the number of degrees of freedom of the scattered field, i.e., the number of orthogonal basis functions required to represent the field within a given precision level.

We now let the $\mathcal{R}$ spatial channels defined by equation (26) evolve over time, assuming independent additive Gaussian noise added at each measurement over time as well. We thus obtain a parallel Gaussian channel, formed by the $\mathcal{R}$ spatial communication modes, whose capacity is well known (cf. [18]) to be

$$C = \sum_{i=1}^{\mathcal{R}} W_t \log \left(1 + \frac{P_i' \ell_i^2}{W_t N_t}\right) \text{ bits per unit time},$$

where $W_t$ is the temporal frequency bandwidth, $N_t$ is the (constant) power spectral density of the additive white Gaussian noise in the time evolution, $W_t N_t = \sigma^2$, and $\{P_i'\}$ are the waterfilling power allocations [4]. If the communication modes are roughly of the same strength, i.e., $\ell_i^2 \approx \frac{1}{\mathcal{R}}$, then the space-time capacity expression (28) simplifies to the sum of $\mathcal{R}$ identical parallel subchannels, given by

$$C = \mathcal{R} W_t \log \left(1 + \frac{P'}{\mathcal{R} W_t N_t}\right) \text{ bits per unit time},$$

where the waterfilling power allocation allocates equal power $P'$ to all channels.
C. A space-time information duality principle

Assuming the sources to be able to excite all $N_0$ degrees of freedom of the field, we can perform the substitution $\mathcal{R} = N_0 = 2\Omega_r \frac{\alpha}{\lambda} = W_s 2\Omega_r$, obtaining

$$C' = 2\Omega_r W_s W_t \log \left( 1 + \frac{P'}{2\Omega_r W_s W_t N_t} \right) \text{ bits per unit time.} \quad (30)$$

Above formula shows the space-time capacity in bits per unit time when the space-time channel is used over time and this channel also exploits the space diversity given by $N_0 = 2\Omega_r W_s$ spatial degrees of freedom.

Similarly, we can obtain a dual formula for the space-time capacity in bits per normalized length, when the space-time channel is used in space as defined by Theorem 2, and this channel also exploits the time diversity of $W_t 2T$ degrees of freedom given by a frequency bandwidth $W_t$. In this case, we need to modify (25) to relate the coefficients $\{\alpha_i\}$ to the communication modes. By assuming as before all the communication modes to be of the same strength, we let

$$\gamma_i' = \alpha_i' + z_i = \ell \beta_i' + z_i,$$

with $\ell^2 = \frac{1}{W_t 2T}$. The waterfilling power allocations allocate equal power $P'$ to all frequency channels and we have

$$C' = 2T W_t W_s \log \left( 1 + \frac{P'}{2T W_t W_s N_s} \right) \text{ bits per unit length.} \quad (31)$$

By comparing equations (30) and (31), we can now state the following space-time information duality principle.

*Information duality principle.* The physical space-time AWGN channel can be operated in time, by summing all the relevant spatial information channels that are related to the (spatial) degrees of freedom of the scattered field through application of the sampling theorem in space. In a dual way the same channel can be operated, in principle, in space (given acceptable error exponents due to the angular resolution limit), by summing all the relevant frequency information channels that are related to the (temporal) degrees of freedom of the scattered field through application of the sampling theorem in time.

It follows that there are two equivalent ways of measuring information capacity: in the first case this is measured in bits per unit time, while in the second case it is measured in bits per units of normalized space. The two information measures are equivalent representations of the same concept of information rate through space and time, in a MIMO system, projected along each of these dimensions. Notice that while the time dimension is present in the first information measure, the second information measure is
a pure number. This is because the bandwidth $W_t$ is measured in Hz, while the spatial bandwidth $W_s$, being normalized to the wavelength, is a pure number. It follows that the two equations (30) and (31) cannot be directly compared. However, we can make both representations independent of dimension by normalizing the bandwidth $W_t$ to the carrier frequency $f_0 = 1/\tau_0$. In this case we measure (30) in bits per time normalized to $\tau_0$, and (31) in bits per length normalized to $\lambda$. A quantitative comparison is now possible, and we see that the two equations coincide, provided that normalized variables $\Omega_r = 2S/r_m$ and $T/\tau_0$ are equal. This means that in our ideal dual-operation of the space-time channel, the codeword spatial length corresponding to the observation domain $2S$, normalized to $r_m$, must be equal to the codeword temporal length $T$, normalized to the period of the temporal carrier frequency $\tau_0$.

V. WIDE-BAND FREQUENCY SPECTRUM

The information theoretic characterization of the radiated field presented in the previous section was based on narrow-band frequency signals. We have assumed that the spatial bandwidth $W_0$ remains constant over the (narrow) frequency band $W_t$ used for transmission. In this way, the information contents along the space and time dimensions could be treated independently and led to the information duality principle. We now want to make a first step in the direction of understanding the properties of the signal received on the manifold $\mathcal{M}$ when the narrow-band assumption is relaxed. In this case, we need to account for the dependence between the spatial band and the frequency band of the radiated wave, generalizing the single frequency treatment of Bucci and Franceschetti [2], [3]. As we shall see in the following, this leads to a variable space-time sampling rate defining the information content of the scattered wave.

Let us start writing the space-time domain representation of the field over the manifold, which is a real function,

\[
e(t, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \ e^{i\omega t} \int_{-\omega a/c}^{\omega a/c} du \ e^{-ius} E(\omega, u),
\]

(32)

where $E(\omega, u)$ is the frequency-wavenumber representation of the received signal, $s$ is the curvilinear coordinate over the manifold, $a$ is the size of the radiating ball, $c$ is the speed of light, and $\omega a/c = W_0$ is the spatial bandwidth of the radiating system at angular frequency $\omega$. Note that in (32) the (real) space-time field representation is obtained by transforming $E(\omega, u)$ with respect to the wavenumber and then inverse transforming with respect to the angular frequency. Letting the inner integral be $\mathcal{F}(\omega, s)$, we note that

\[
e(t, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \ e^{i\omega t} \mathcal{F}(\omega, s)
\]

(33)
is real if \( F(-\omega, s) = F^*(\omega, s) \). Hence, we can write

\[
e(t, s) = \frac{1}{\pi} \Re \left\{ \int_0^\infty d\omega \ e^{i\omega t} \int_{-\omega_0/c}^{\omega_0/c} du \ e^{-ius} E(\omega, u) \right\}.
\]

(34)

We now assume the function \( E(\omega, u) \) to be frequency band-limited to the interval \([\omega_0 - \Delta \omega/2, \omega_0 + \Delta \omega/2]\). It follows that

\[
e(t, s) = \frac{1}{\pi} \Re \left\{ \int_{\omega_0 - \Delta \omega/2}^{\omega_0 + \Delta \omega/2} d\omega \ e^{i\omega t} \int_{-\omega_0/c}^{\omega_0/c} du \ e^{-ius} E(\omega, u) \right\}.
\]

(35)

Above equation is the starting point for the analysis of the two cases of narrow-band and wide-band spectra.

A. Case 1. Narrow-band

Let us consider first radiating a pure sinusoidal signal at angular frequency \( \omega_0 \). That is, a time domain signal of the type \( \cos(\omega_0 t) \). In this case, for \( \omega > 0 \), we have

\[
E(\omega, u) = \pi \delta(\omega - \omega_0).
\]

(36)

When no spatial modulation has been enforced on \( E(\omega, u) \), we have from (35),

\[
e(t, s) = \frac{1}{\pi} \Re \left\{ \int_{\omega_0 - \Delta \omega/2}^{\omega_0 + \Delta \omega/2} d\omega \ e^{i\omega t} \delta(\omega - \omega_0) \frac{2\omega_0 a}{c} \ \text{sinc}\left(\frac{\omega_0 s}{c}\right) \right\}
\]

\[\]

\[= \frac{2\omega_0 a}{c} \ \text{sinc}(\omega_0 t) \ \text{sinc}\left(\frac{\omega_0 a s}{c}\right) \]

\[= 2W_0 \cos(\omega_0 t) \ \text{sinc}(W_0 s),
\]

(37)

where we have adopted the convention \( \text{sinc}(x) = (\sin x)/x \). Above equation shows that the spatial signal is concentrated around \( s = 0 \), as expected, because no spatial modulation has been enforced on \( E(\omega, u) \). Clearly, we can move the maximum of the signal in space by appropriate spatial modulation, thus obtaining a different spatial illumination of the type \( \text{sinc}\left[\frac{\omega_0 a (s - s_0)}{c}\right] \). In this way, two sufficiently distinct \( \text{sinc}(\cdot) \) patterns can be singled out provided that the spacing \( \Delta s \) between them corresponds to at least half a lobe of the \( \text{sinc}(\cdot) \) function, that is \( \frac{\omega_0 a}{c} \Delta s = \frac{\pi a}{W_0} \Delta s > \pi \), i.e.,

\[
\Delta s > \frac{\lambda}{2a} = \frac{\pi}{W_0}.
\]

(38)

This time-domain analysis is consistent with the corresponding frequency domain analysis of Section III, leading to (18), and with antenna theory, which sets a limitation on the attainable resolution of antennas of dimension \( 2a \). While the frequency domain analysis leading to (18) is based on the application of the
sampling theorem in space, the time-domain viewpoint leading to the equivalent (38) sets the resolution limit on the manifold as the angular separation needed to obtain spots over the manifold where each receiving antenna can detect essentially a single signal. Notice that while in (38) ∆s is normalized to \( r_m \), this is not the case in (18) where the normalizing factor appears explicitly.

It is interesting now to consider radiating a pure sinusoidal signal of spatial wavenumber \( u_0 \), i.e., a space domain signal of the type \( \cos(u_0s) \). In this case, for \( u > 0 \), we have

\[
E(\omega, u) = \pi \delta(u - u_0). 
\]

(39)

The inner integral in (35) can then be evaluated to yield

\[
\int_{-\omega_0/c}^{\omega_0/c} du e^{-iu_0s} E(\omega, u) = \begin{cases} \pi e^{-iu_0s}, & \text{if } -\frac{\omega_0}{c} < u_0 < \frac{\omega_0}{c}, \\ 0, & \text{otherwise}. \end{cases} 
\]

(40)

Clearly, in order to generate a sinusoid at an arbitrary spatial frequency, we need to let the radiating system dimension \( a \to \infty \). In this case, (35) can be evaluated as follows,

\[
e(t, s) = \text{Re} \left\{ \int_{\omega_0 - \Delta \omega/2}^{\omega_0 + \Delta \omega/2} d\omega e^{i\omega t} e^{-iu_0s} \right\} \\
= \text{Re} \left\{ \Delta \omega \sin\left(\frac{\Delta \omega t}{2}\right) e^{i(\omega t - u_0s)} \right\} \\
= \Delta \omega \cos(\omega_0 t - u_0s) \sin\left(\frac{\Delta \omega t}{2}\right). 
\]

(41)

By analogy with (37) we see that now the time signal is concentrated around \( t = 0 \) and that two distinct time signals can be singled out, by performing appropriate frequency modulation, provided that the time spacing \( \Delta t \) between them satisfies the Nyquist resolution limit,

\[
\Delta t > \frac{2\pi}{\Delta \omega}. 
\]

(42)

We conclude that in the narrow-space and narrow-frequency case, space and time signals are decoupled and the appropriate sampling rate is determined by the frequency and space bandwidths respectively.

Next, we consider the case of wide band signals, where it turns out that space and time signals cannot be decoupled. Since the spatial bandwidth is a function of the radiated frequency, the amount of diversity available in space depends on a specific position in frequency, and similarly the amount of diversity available in time depends on a specific position in space.

B. Case 2. Wide-band

We now want to consider a radiated waveform of frequency spectrum \( \Delta \omega \) and space spectrum \( W_0(\omega) \) that are not negligible. For convenience, we assume the frequency-wavenumber representation of the
received signal $E(\omega, u)$ to be a constant $E_0$ over both the frequency and the wavenumber intervals of interest. The inner integral of (35) can then be written as follows,

$$F(\omega, s) = E_0 \int_{-\omega_a/c}^{\omega_a/c} du e^{-ius} = \frac{E_0}{is} \left( e^{i\omega as/c} - e^{-i\omega as/c} \right).$$

(43)

Computing the outer integral we have that

$$\int_{\omega_0 - \Delta \omega/2}^{\omega_0 + \Delta \omega/2} d\omega e^{i\omega t} F(\omega, s) = \frac{E_0 \Delta \omega}{is} \left\{ e^{i\omega_0 (t + as/c)} \sin \left[ \frac{\Delta \omega}{2} \left( t + \frac{as}{c} \right) \right] \right. $$

$$- e^{i\omega_0 (t - as/c)} \sin \left[ \frac{\Delta \omega}{2} \left( t - \frac{as}{c} \right) \right] \left\}. \right.$$  

(44)

To obtain $e(t, s)$ we finally multiply by $1/\pi$ and take the real part, obtaining

$$e(t, s) = \frac{E_0 \Delta \omega}{\pi s} \left\{ \sin \left[ \omega_0 \left( t + \frac{as}{c} \right) \right] \sin \left[ \frac{\Delta \omega}{2} \left( t + \frac{as}{c} \right) \right] \right. $$

$$- \sin \left[ \omega_0 \left( t - \frac{as}{c} \right) \right] \sin \left[ \frac{\Delta \omega}{2} \left( t - \frac{as}{c} \right) \right] \left\}. \right.$$  

(45)

We now compare (45) with (37) and (41). It is clear that while time and space are decoupled in the narrowband case, a more complex relationship arises in the wideband case. In the frequency narrowband case (37) the spatial resolution is dictated by the amount of shift of a single sinc($\cdot$) function translated in space; similarly in the space narrowband case (41) the time resolution is dictated by the amount of shift of a single sinc($\cdot$) function translated in time. Instead, time and space appear now linked to each other in (45), which is a consequence of the fact that the spatial band $W_0(\omega) = \omega a/c$ now varies with the frequency $\omega$ over the interval $[\omega_0 - \Delta \omega/2, \omega_0 + \Delta \omega/2]$. A somewhat clearer picture arises if we elaborate on (45) and rewrite it as follows,

$$e(t, s) = \frac{E_0 \Delta \omega}{\pi} \left[ F^-(t, s) + F^+(t, s) \right],$$

(46)

where

$$F^-(t, s) = \frac{\Delta \omega a}{c} \sin(\omega_0 t) \cos \left( \frac{\omega_0 as}{c} \right) \left\{ \frac{\sin \left[ \frac{\Delta \omega}{2} \left( t + \frac{as}{c} \right) \right] - \sin \left[ \frac{\Delta \omega}{2} \left( t - \frac{as}{c} \right) \right]}{\Delta \omega \frac{as}{c}} \right\},$$

(47)

$$F^+(t, s) = \frac{\omega_0 a}{c} \cos(\omega_0 t) \sin \left( \frac{\omega_0 as}{c} \right) \left\{ \sin \left[ \frac{\Delta \omega}{2} \left( t + \frac{as}{c} \right) \right] + \sin \left[ \frac{\Delta \omega}{2} \left( t - \frac{as}{c} \right) \right] \right\}. $$

(48)

These equations clearly show that the signal is the superposition of two quadrature components and allow comparison of the corresponding envelopes. We notice that the frequency narrowband case expressed by (37) is immediately recovered by letting $\Delta \omega \to 0$ while keeping $E_0 \Delta \omega$ constant, and noticing that

$$\lim_{\Delta \omega \to 0} F^-(t, s) = \lim_{\Delta \omega \to 0} \frac{\Delta \omega a}{c} \sin(\omega_0 t) \cos \left( \frac{\omega_0 as}{c} \right) \left\{ \frac{\partial}{\partial \omega_0 t} \sin \left( \frac{\Delta \omega t}{2} \right) \right\} = 0.$$  

(49)

The spatial narrowband case cannot be directly recovered by letting $a \to \infty$, because the limit cannot be carried inside the integral (35), while the proper steps to be followed are the ones leading to (37).
VI. Conclusion

In MIMO systems communication is performed through EM waves and the concept of information transmission is related to the amount of diversity EM waves can carry. Such diversity lies in two different dimensions: time and space. The classical view of Shannon’s information theory typically considers only the time dimension along with its transformed counterpart: the frequency spectrum. We have shown that Shannon’s theory can also be applied to the space dimension which, analogous to time, becomes a capacity bearing object. The concept of transformed space domain and space bandwidth has been introduced and an information duality principle has shown that space and time can naturally be treated as the dual of one another. In the wide-band transmission regime a much more complex scenario arises. In this case, time and space are linked together, as the space bandwidth depends on the radiated frequency. Our analysis appears to be open to several refinements. In particular, characterization of the capacity in the frequency wide-band regime and showing that although space and time are not decoupled, information is conserved in space-time are interesting open problems.

VII. Acknowledgement

The authors wish to acknowledge enlightening conversations they had on the subject with Prof. Giorgio Franceschetti, to whom this work is dedicated.

References


Fig. 1. Geometry of the scattering problem. The transmitters (denoted by crosses) and the scatterers (denoted by ovals) are assumed to be enclosed within a ball $B$ of radius $a$, and the observation manifold $M$ is located in the far field ($r_m \gg a$). The manifold is measured in curvilinear coordinates of variable $s$.

<table>
<thead>
<tr>
<th>Carrier frequency $f_c$ (in MHz)</th>
<th>$W_0$</th>
<th>$N_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>900</td>
<td>$6\pi \times 10^3$</td>
<td>60</td>
</tr>
<tr>
<td>1900</td>
<td>$12.67\pi \times 10^3$</td>
<td>126</td>
</tr>
<tr>
<td>2400</td>
<td>$16\pi \times 10^3$</td>
<td>160</td>
</tr>
</tbody>
</table>

Table I: Spatial bandwidth and number of degrees of freedom for Example 2.1.
Fig. 2. Geometric interpretation of the number of degrees of freedom of the scattered field.

### Table II: Spatial bandwidth and number of degrees of freedom for Example 2.2.

<table>
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<th>$W_0$</th>
<th>$N_0$</th>
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</thead>
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### Table III: Table of values for Example 4.1.

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<th>Carrier frequency $f_c$ (in MHz)</th>
<th>$R_{\text{max}}/C$ Example 2.1</th>
<th>$R_{\text{max}}/C$ Example 2.2</th>
</tr>
</thead>
<tbody>
<tr>
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